HENSTOCK- KURZWEIL INTEGRAL ON [a,b]

THESIS

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OF MALANG
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MOTTO

Humans not only consists of the body, but inside there hearts and souls are adorned selfhood

DEDICATION

This Thesis dedicated for
My Beloved Father Abd.Rahman,
My Beloved Mother Alm. Masiyah And
My Beloved Brother Wafdi Faiz

STATEMENT AUTHENTICITY OF TEXT

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Malang, June 25th 2010

Siti Nurul Afiyah.

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TABLE OF SYMBOL

$\subset$	Subset
$\Re$	Real Number
$\in$	Element
$\forall$	For Every
δ	Delta
$<$	Less Than
$\leq$	Less Than or Equal to
$>$	Greater than
$\geq$	Greater than or Equal to
ε	Epsilon
$\lvert ... \rvert$	Absolute value
$\cup$	Union
$\cap$	Intersection
[..]	Closed Interval
(..)	Open interval
[..)	Closed Open interval
(..]	Open closed interval
Sup	Supremum
Inf	Infimum
$\notin$	Not element
ξ	Xi
B	Ball

ABSTRACT

Afiyah, Siti Nurul. 2010. **Henstock-Kurzweil Integral on [a,b].**Thesis. Mathematics Department Faculty of Science and Technology State Islamic University Maulana Malik Ibrahim of Malang
Advisor : Hairur Rahman, M.Si.
Dr. Munirul Abidin, M.Ag.

Key words: Riemann integral, $\delta - fine$ partition, Henstock-Kurzweil integral.

The theory of the Riemann integral was not fully satisfactory. Many important functions do not have a Riemann integral. So, Henstock and Kurzweil make the new theory of integral. From the background, the writer will be research about Henstock-Kurzweil integral and also theorems of Henstock- Kurzweil Integral. Henstock- Kurzweil Integral is generalized from Riemann integral.

In this case the writer uses research methods literature or literature study carried out by way explore, observe, examine and identify the existing knowledge in the literature

In this thesis explain about partition which used in Henstock- Kurzweil Integral, definition and some property of Henstock- Kurzweil Integral. And some properties of Henstock- Kurzweil integral as follows: value of the Henstock-Kurzweil integral is unique, linearity of the Henstock-Kurzweil integral, Additivity of the Henstock-Kurzweil integral, Cauchy criteria, nonnegativity of Henstock-Kurzweil integral and primitive function.

CHAPTER I
INTRODUCTION

1.1 Background of The Study

Some of the history of natural science is a record of continuous human efforts to formulate concepts and elements in the field of science to be elaborated into the real world.

Speaking about science, Al-Qur'an has given the key to human is knowledge about the world and the here after and provide tools for finding and researching all things in order to uncover and learn the magic of both worlds. (Rahman, 1992:12).

No doubt that the Qur'an with recommendations to think that repeating a few times to make studies and research activities in various fields as an obligation for Muslims. Therefore, Islam ordered the human to worship and think.

Human beings have been created with an excess of reason and mind, has a very important role to be able to explore and utilize all forms of His creation. As explained in the Qur'an. With all the human excesses contribute to developing science. Furthermore through studies and research activities, human are expected to understand the truth of Qur'an. In Islam a Muslim is obliged to seek knowledge even if the search for knowledge that far.

In QS Al-Kahfi verse 109 Allah SWT says:

قُل لَّوۡ كَانَ ٱلۡبَحۡرُ مِدَادٗا لِّكَلِمَٰتِ رَبِّي لَنَفِدَ ٱلۡبَحۡرُ قَبۡلَ أَن تَنفَدَ كَلِمَٰتُ رَبِّي وَلَوۡ جِئۡنَا بِمِثۡلِهِۦ مَدَدٗا ١٠٩

"Say: Had oceans become ink for (write) sentences sovereign, really screwed sea would before discharged (written) sentences sovereign, although we bring additional much (too)". (QS Al-Kahfi:109)

The verse explains that should the human understanding the obligation to study and learn it. In studying science not only armed intellects ability but also need supported simultaneously with emotional and spiritual ability. So if someone have understand a science, hence person can convey science that been owned, to other person with good method so what conveyed easily understood by others. As God commanded the Prophet to convey to people about a science. Word situates on QS Al-Maidah verse 99;

مَّا عَلَى ٱلرَّسُولِ إِلَّا ٱلۡبَلَٰغُۗ وَٱللَّهُ يَعۡلَمُ مَا تُبۡدُونَ وَمَا تَكۡتُمُونَ ٩٩

"*Obligation the Prophet only to convey, and Allah knows what you reveal and what you hide*." (QS Al-Maidah: 99)

In life on earth, human beings are not free from various problems. The problems concerning various aspects, which in its solution required an understanding through a method. Mathematics is one branch of science that underlies many other knowledge and always face a variety of increasingly complex phenomena, so important to learn. Mathematics is a tool to simplify the presentation and understanding of the problem. In the discussion of mathematics, a problem can be simplified to be presented, understood, analyzed and solved. For this purpose, first search the principal problem. Then, make formula or mathematical model of it (Purwanto, 1998: 1).

Learn mathematics in accordance with the *ulul albab* paradigm , not only armed intellect ability, but also needs supported simultaneously with the emotionally and spiritually ability. Deductive thinking patterns and logical in mathematics will also depend on the intuitive and imaginative ability, and develop a rationalist approach, empirical and logical. While the deductive thinking pattern is a pattern of thinking which is based on those truths which in general have proven true (Abdussakir, 2007: 24).

As in the word of God in QS Al-Imran: 191:

ٱلَّذِينَ يَذۡكُرُونَ ٱللَّهَ قِيَٰمٗا وَقُعُودٗا وَعَلَىٰ جُنُوبِهِمۡ وَيَتَفَكَّرُونَ فِي خَلۡقِ ٱلسَّمَٰوَٰتِ وَٱلۡأَرۡضِ رَبَّنَا مَا خَلَقۡتَ هَٰذَا بَٰطِلٗا سُبۡحَٰنَكَ فَقِنَا عَذَابَ ٱلنَّارِ ١٩١

"(They are) those who remember Allah, standing, sitting or lying down and they think about the creation of the heavens and the earth, (saying):" Oh my God, Nor You created this in vain, Glory to Thee, And save us from the torment of hell." (QS Al-Imran: 191)

In accordance with the objectives of mathematics instruction that trains in a systematic way of thinking, logical, analytical and critical. And in mathematics there is one area where it is required in learning to think analytically. Namely is the field of analysis. One theory that is learned in the field of analysis is the theory of integral. Like other sciences are integral theory in mathematics is a deductive science and still grow like other sciences, both in terms of theory and application.

We have already mentioned the developments, during the 1630's, by Fermat and Descrates leading to analytic geometry and the theory of the derivatives. However, the subject we know as calculus did not begin to take shape until the late 1660's when Issac Newton (1642-1727) created his theory of fluxions and invented the method of inverse tangents to find areas under curves. The reversal of the process for finding tangent lines to find areas was also discovered in the 1680's by Leibniz(1646-1716), who was unaware of Newton unpublished work and who arrived at the discovery by a very different route. Leibniz introduced the terminology calculus differential and calculus integral, since finding tangents lines involved differences and finding areas involved summations. Thus they had discovered that integration, being a process of summation, was inverse to the operation of differentiation.

During a century and a half of development and refinement of techniques, calculus consisted of these paired operations and their applications, primarily to physical problems. In the 1850s, Bernhard Riemann (1826-1866) adopted a new and different viewpoint. He separated the concept of integration from its companion, differentiation, and examined the motivating summation and limit process of finding areas by itself. He broadened the scope by considering all functions on an interval for which this process of integration could be defined: the class of integrable functions. The fundamental Theorem of calculus became a result that held only for a restricted set of integrable functions. The viewpoint of Riemann led others to invent other integration theories, the most significant being Lebesgue's theory of integration.

The theory of the Riemann integral was not fully satisfactory. Many important functions do not have a Riemann integral even after we extend the class of integrable functions slightly by allowing "improper" Riemann integrals. For example Characteristic function.

In 1957, the Czech mathematician Jaroslav Kurzweil discovered a new definition of this integral elegantly similar in nature to Riemann's original definition which he named the gauge integral; the theory was developed by Ralph Henstock. Due to these two important mathematicians, it is now commonly known as Henstock-Kurzweil integral. The simplicity of Kurzweil's definition made some educators advocate that this integral should replace the Riemann integral in introductory calculus courses, but this idea has not gained traction.

1.2 Statement of Problem

Concerning the background of the study, the writer formulates the statement of the problems as follows:

1. How does the concept δ-fine partition of Henstock-Kurzweil Integral?
2. How does the definition of Henstock-Kurzweil Integral?
3. How does the fundamental properties of Henstock-Kurzweil Integral?

1.3 Objectives of Study

As it has been stated in the problem of the study, this study 6 conducted to find out:

1. To know the concept δ-fine partition of Henstock-Kurzweil Integral.
2. To know the definition of Henstock-Kurzweil Integral.
3. To know the fundamental properties of Henstock-Kurzweil Integral.

1.4 Significance of Study

The Significance of Study is to:

1. Writer

The significance of study for the author is as follows:

a. As form of development of the science researchers who have gained during college.

b. As reference material in increasing knowledge about Henstock-Kurzweil integral on $[a,b]$.

2. Reader

The significance of study to the reader is as follows:

a. As a starting point for discussion that could continue or be developed.

b. As a vehicle of increasing scientific treasure

3. Institutions

Adding library materials at institutions especially in the faculty of science and technology the State Islamic University Maulana Malik Ibrahim of Malang can be used as a means of developing depth of knowledge especially in mathematics department.

1.5 Research Method

For the purposes of data analysis, the researchers need some supporting data. In this case the author uses research methods literature or literature study carried out by way explore, observe, examine and identify the existing knowledge in the literature (literature sources, reference books or other research results).

1.7 Systematic review

To make the reader understand this paper, the author of this paper divides into four chapters as follows:

CHAPTER I Introduction.

In this chapter, we explained about the background of the problem, formulation of the problem, the problem definition, objectives and benefits and methods of research.

CHAPTER II Review of the related Literature.

Study of the theory which contains the basic concepts and theorems that support the discussion of Henstock-Kurzweil integral and some theorems and discussion of the examples used as a reference or comparison in the discussion

CHAPTER III Discussion.

In this chapter presented the results of studies that include a definition of the concept of $\delta - fine$ partition, the definition of Henstock-Kurzweil integral and fundamental properties of Henstock-Kurzweil integral.

CHAPTER IV Conclusion.

Contains conclusions and suggestions as a follow-up for readers who want to develop a discussion about Henstock-Kurzweil integral on $[a,b]$.

CHAPTER II
REVIEW OF THE RELATED LITERATURE

2.1 Supremum and Infimum

We start with a straightforward definition similar to many others in this course. Read the definitions carefully, and note the use of $\leq$ and $\geq$ here rather than $<$ and $>$.

Definition 2.1.1 (Bartle, 1994: 42)

Let S be a subset of $\Re$

1. A number $u \in \Re$ is said to be an upper bound of S if $s \leq u$ for all $s \in S$
2. A number $w \in \Re$ is said to be a lower bound of S if $w \leq s$ for all $s \in S$

Definition 2.1.2 (Bartle, 1994: 43)

Let S be a subset of $\Re$.

1. If S is bounded above, then an upper bound u is said to be *supremum* (or a least upper bound) of S if no number smaller than u is an upper bound of S.
2. If S is bounded below, then a lower bound w is said to be *infimum* (or a greatest lower bound) of S if no number greater than w is a lower bound of S.

Example 2.1.3

The set $S := \{x : 0 \leq x \leq 1\}$ clearly has 1 for an upper bound. We prove 1 is the supremum as follows. If $v < 1$, there exist an element $s' \in S$ such that $v < s'$. Hence v is not an upper bound of S and since v is an arbitrary number $v < 1$, we conclude that $\sup S = 1$. Similarly one can show that $\inf S = 0$. N 9 that both sup S and inf S are contained in the set S.

2.2 Limit of Function

The essence of the concept of limit for real valued functions of a real variable is this: if L is a real number, then $\lim_{x \to x_0} f(x) = L$ means that the value *f(x)* can be made as close to L as we wish by taking x sufficiently close to x_0. This is made precise in the following definition.

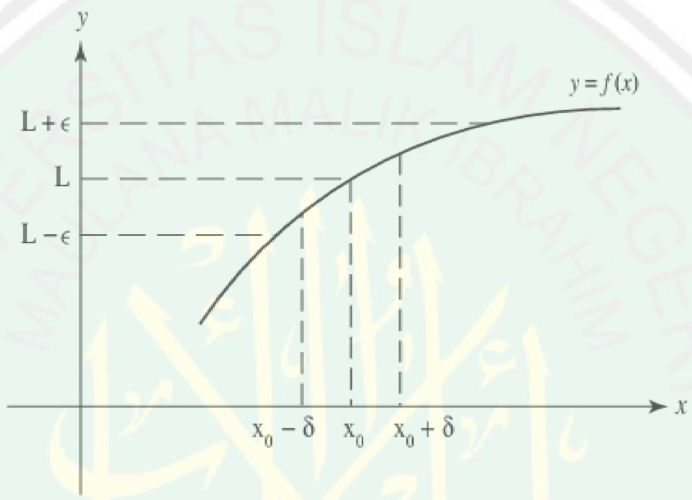

Figure 2.1 The limit of f at x_0 is L

Definition 2.2.1 (William, 2009: 34)

We say that *f(x)* approaches the limit L as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = L$$

If f is defined on some deleted neighborhood of x_0, and for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

If

$$0 < |x - x_0| < \delta$$

Example 2.2.2

If c and x are arbitrary real numbers and $f(x) = cx$, then

$$\lim_{x \to x_0} f(x) = cx_0$$

To prove this, we write

$$|f(x) - cx_0| = |cx - cx_0| = |c||x - x_0|$$

If $c \neq 0$, this yields

$$|f(x) - cx_0| < \varepsilon$$

If

$$|x - x_0| < \delta$$

Where δ is any number such that $0 < \delta \leq \varepsilon / |c|$. If c = 0, then $f(x) - cx_0 = 0$ for all x. so $|f(x) - cx_0| < \varepsilon$ holds for all x.

Example 2.2.3

$\lim_{x \to 2} x^3 = 8$. To see this, one can manipulate Lipschitz condition, but it is perhaps simpler to write

$$|x^3 - 8| = |x - 2||x^2 + 2x + 4|$$

Clearly we can make the $|x - 2|$ term small by making x close to 2, but we do have to make sure that the $|x^2 + 2x + 4|$ term is bounded. However, if we make x within say 1 to 2, then wont exceed 3, so $|x^2 + 2x + 4| < 19$. Thus, choose $\delta < \min(\varepsilon / 19, 1)$ so $|x^3 - 8| = |x - 2||x^2 + 2x + 4| < (\varepsilon / 19)(19) = \varepsilon$

2.3 Compact Sets

Definition 2.3.1(Bartle, 1994: 355)

A subset K of R is said to be compact if every open cover of K has a finite sub cover.

In other words, a set K is compact if, whenever it is contained in the union of a collection $\mathcal{G}=\{G_\alpha\}$ of open sets in R , then it is contained in the union of some finite number of sets in $\mathcal{G}$.

Theorem 2.3.2 (Bartle, 1994: 357)

If K is a compact subset of R, then K is closed and bounded.

Proof: We shall first show that K is bounded. For each $m \in N$, let $H_m := (-m, m)$. Since each H_m is open and since $K \subseteq \bigcup_{m=1}^{M} H_m = R$, we see that the collection $\{H_m : m \in N\}$ is an open cover of K. since K is compact, this collection has a finite sub cover , so there exists $M \in N$ such that,

$$K \subseteq \bigcup_{m=1}^{M} H_m = H_M = (-M, M)$$

Therefore K is bounded, since it is contained in the bounded interval $(-M, M)$.

We show that K is bounded, by showing that its complement $u = \mathscr{b}(K)$ is open. To do so, let $u = \mathscr{b}(K)$ be arbitrary and for each $n \in N$, we let $G_n := \{y \in R : |y - u| > {}^{1}/_{n}\}$. Since $u \notin K$, we have $K \subseteq \bigcup_{n=1}^{\sim} G_n$. Since K is compact, there exists $m \in N$ such that

$$K \subseteq \bigcup_{n=1}^{\sim} G_n = G_m$$

Now it follows from this that $K \cap (u - 1/m, u + 1/m) = \emptyset$, so that $(u - 1/m, u + 1/m) \in \mathcal{b}(K)$. but since u was an arbitrary point in $\mathcal{b}(K)$, we infer that $\mathcal{b}(K)$ is open.

2.4 Continuity

Definition 2.4.1 (William, 2009: 54)

a) We say that f is continuous at x_0 if f is defined on an open interval (a,b) containing x_0 and $\lim_{x \to x_0} f(x) = f(x_0)$.

b) We say that f is continuous from the left at x_0 if f is defined on an open interval (a, x_0) and $f(x_0-) = f(x_0)$.

c) We say that f is continuous from the right at x_0 if f is defined on an open interval (x_0, b) and $f(x_0+) = f(x_0)$

Theorem 2.4.2 (William, 2009: 54)

a) A function f is continuous at x_0 if and only if f is defined on an open interval (a,b) containing x_0 and for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon$$

Whenever $|x - x_0| < \delta$

b) A function f is continuous from the right at x_0 if and only if f is defined on an open interval $[x_0, b)$ and for each $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ holds whenever $x_0 \le x \le x_0 + \delta$

c) A function *f* is continuous from the left at x_0 if and only if *f* is defined on an open interval $(a, x_0]$ and for each $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ holds whenever $x_0 - \delta \le x \le x_0$

Example 2.4.3

let f be defined on $[0,2]$ by

$$f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ x+1, & 1 \le x \le 2 \end{cases}$$

(Figure 2. 5); Then

$$\begin{aligned} f(0+) &= 0 = f(0), \\ f(1-) &= 1 \ne f(1) = 2, \\ f(1+) &= 2 = f(1), \\ f(2-) &= 3 = f(2), \end{aligned}$$

Therefore *f* is continuous from the right at 0 and 1 continuous the left at 2, but not at 1. If $0 < x, x_0 < 1$ then

$$\begin{aligned} |f(x) - f(x_0)| &= |x^2 - x_0^2| = |x - x_0| + |x + x_0| \\ &< 2|x - x_0| < \varepsilon \ \ if \ \ |x - x_0| < \varepsilon/2 \end{aligned}$$

Hence, f is continuous at each x_0 in $(0,1)$. If $1 < x, x_0 < 2$ then

$$\begin{aligned} |f(x) - f(x_0)| &= |(x+1) - (x_0 + 1)| = |x - x_0| \\ &< \varepsilon \ \ if \ \ |x - x_0| < \varepsilon \end{aligned}$$

Hence f is continuous at each x_0 in $(1,2)$.

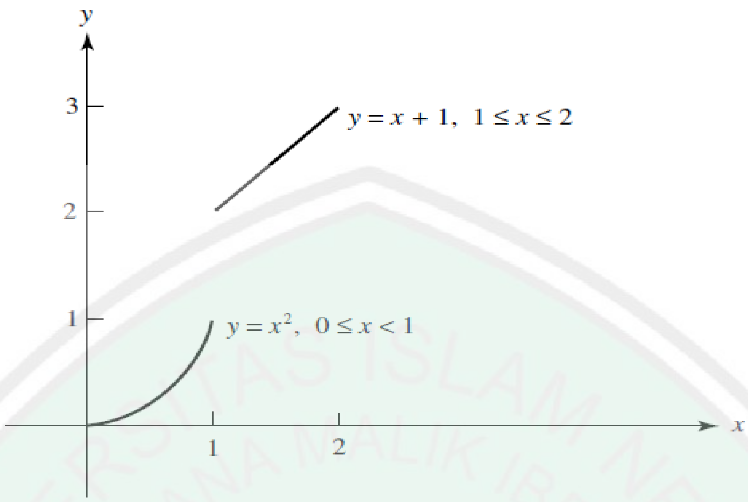

Figure 2.2 Function *f(x)* Defined on $[0,2]$

Definition 2.4.4 (William, 2009: 55)

A function *f* is continuous on an open interval (a,b) if it is continuous at every point in (a,b). if, in addition.

$$f(b-) = f(b) \text{ Or } f(a+) = f(a)$$

Then f is continuous on $(a,b]$ *or* $[a,b)$, respectively, if *f* continuous on (a,b) and $f(b-) = f(b)$ Or $f(a+) = f(a)$ both hold, then *f* is continuous on $[a,b]$.

Example 2.4.5

Let $f(x) = \sqrt{x}, 0 \le x < \infty$. then

$$|f(x) - f(0)| = \sqrt{x} < \varepsilon \quad if \quad 0 \le x < \varepsilon^2,$$

So $f(0+) = f(0)$. If $x_0 > 0$ *and* $x \ge 0$, then

$$|f(x) - f(x_0)| = \left|\sqrt{x} - \sqrt{x_0}\right| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}}$$

$$\le \frac{|x - x_0|}{\sqrt{x_0}} < \varepsilon \ \ if \quad |x - x_0| < \varepsilon\sqrt{x_0}$$

So $\lim_{x \to x_0} f(x) = f(x_0)$. . Hence, *f* is continuous on $[0,\infty)$

Definition 2.4.6 (William, 2009: 56)

A function *f* is piecewise continuous on $[a,b]$ if

a) $f(x_0+)$ exist for all x_0 in $[a,b)$;

b) $f(x_0-)$ exist for all x_0 in $(a,b]$;

c) $f(x_0+) = f(x_0-) = f(x_0)$ for all but finitely many points x_0 in (a,b).

If c) fails to hold at some x_0 in (a,b), *f* has a jump discontinuity at x_0.

Also, *f* has a jump discontinuity at a if $f(a+) \neq = f(a)$ or at b if $f(b-) \neq f(b)$.

Example 2.4.7

The function

$$f(x) = \begin{cases} 1, & x = 0 \\ x, & 0 < x < 1 \\ 2, & x = 1 \\ x, & 1 < x \leq 2 \\ -1, & 2 < x < 3 \\ 0, & x = 3, \end{cases}$$

(Figure 2.6) is the graph of piecewise continuous function on [0,3], with jump discontinuities at $x_0 = 0,1,2$ and 3.

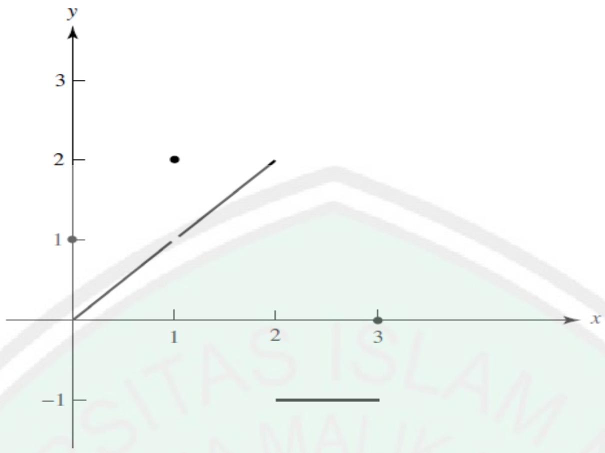

Figure 2.3 The graph of piecewise continuous function on [0,3], with jump discontinuities at $x_0 = 0,1,2$ and 3.

2.5 Uniform Continuity

Definition 2.5.1 (Bartle, 1994: 162)

Let $A \subseteq R$, let $f : A \rightarrow R$,we say that *f* is uniformly continuous on *A* if for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $x,\ u \in A$ are any number satisfying , then $|f(x) - f(u)| < \varepsilon$.

Theorem 2.5.2(Bartle, 1994: 162)

If f is continuous on a closed interval $[a,b]$, then *f* is uniformly continuous on $[a,b]$.

Proof:

Suppose that $\varepsilon > 0$. Since f is continuous on $[a,b]$, for each t in $[a,b]$ there is a positive number δ_t such that

$$|f(x) - f(t)| < \frac{\varepsilon}{2} \qquad \text{if } |x - t < 2\delta_t| \qquad \text{and} \quad x \in [a,b]$$

If $I_t = (t-\delta_t, t+\delta_t)$, the collection

$$H = \{I_t | t \in [a,b]\}$$

Is an open covering of $[a,b]$. Since $[a,b]$ is compact, the Heine-Borel theorem implies that there are finitely many points $t_1, t_2, ..., t_n$ an $[a,b]$ such that $I_{t_1}, I_{t_2}, ..., I_{t_n}$ cover $[a,b]$.

Now define

$$\delta = \min\{\delta_{t_1}, \delta_{t_2}, ..., \delta_{t_n}\}$$

We will show that if

$$|x-x'| < \delta \text{ and } x, x' \in [a,b]$$

Then

$$|f(x) - f(x')| < \varepsilon$$

From the triangle inequality.

$$|f(x) - f(x')| = |(f(x) - f(t_r)) + (f(t_r) - f(x'))|$$

$$\leq |(f(x) - f(t_r))| + |(f(t_r) - f(x'))|$$

Since $I_{t_1}, I_{t_2}, ..., I_{t_n}$ cover $[a,b]$, x must be in one of these intervals. Suppose that $x \in I_{t_r}$;

That is,

$$|x - t_r| < \delta_{t_r}$$

With $t = t_r$,

$$|f(x) - f(t_r)| < \frac{\varepsilon}{2}.$$

Such that,

$$|x' - t_r| = |(x' - x) + (x - t_r)| \le |x' - x| + |x - t_r| < \gamma + \delta_{t_r} \le 2\delta_{t_r}.$$

Therefore with $t = t_r$ and x replaced by x' implies that

$$|f(x') - f(t_r)| < \frac{\varepsilon}{2}.$$

This imply that $|f(x) - f(x')| < \varepsilon$.

Definition 2.5.3 (Bartle, 1994: 163) **(Lipschitz Functions)**

Let $A \subseteq R$, let $f : A \to R$. If there exist a constant $K > 0$ such that

$$|f(x) - f(u)| \le K|x - u|$$

For all $x, u \in A$, then *f* is said to be a **Lipschitz Functions** on *A*

Theorem 2.5.4 (Bartle, 1994: 164)

If $f : A \to R$ is a Lipschitz Functions, then *f* is uniform continuous on *A*.

Proof:

If the a Lipschitz conditions satisfied with constant *K*, then given $\varepsilon > 0$, we can take $\delta := \varepsilon / K$. **If** $x, u \in A$ satisfy $|x - u| < \delta$, then

$$|f(x) - f(u)| < K \cdot \frac{\varepsilon}{K} = \varepsilon$$

Therefore *f* is uniformly continuous on A.

Example 2.5.5

If $f(x) = x^2$ on $A = [0, b]$, where *b* is a positive constant, then

$$|f(x) - f(u)| = |x + u||x - u| \le 2b|x - u|.$$

For all x, u in $[0,b]$. Thus f satisfies a lipschitz condition with constant $K = 2b$ on A. and therefore f is uniformly continuous on A. of course , since f is continuous and A is a closed bounded interval, this can also be deduced from the Uniform continuity Theorem.

2.6 Upper and Lower Integral

Definition 2.6.1 (William, 2009: 120)

If f is bounded on $[a,b]$ and $P\{x_0, x_1,..., x_n\}$ is a partition of $[a,b]$, let

$$M_j = \sup_{x_{j-1} \le x \le x_j} f(x)$$

And

$$m_j = \inf_{x_{j-1} \le x \le x_j} f(x)$$

The upper sum of f over P is

$$S(P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1})$$

And the upper integral of f over $[a,b]$, denoted by

$$\overline{\int_a^b} f(x)dx$$

Is the infimum of all upper sums. The lower sum of f over P is

$$s(P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1})$$

And the lower integral of f over $[a,b]$, denoted by

$$\underline{\int_a^b} f(x)dx$$

Is the supremum off all lower sums.

Example 2.6.2

Let

$$f(x)=\begin{cases}0 & \text{if } x \text{ is irational}\\ 1 & \text{if } x \text{ is rational}\end{cases}$$

And $P=\{x_0,x_1,...,x_n\}$ be a partition of $[a,b]$, since every interval contains both rational and irrational numbers.

$$m_j=0 \text{ and } M_j=1,\ \ 1\le j\le n.$$

Hence,

$$S(P)=\sum_{j=1}^{n}1(x_j-x_{j-1})=b-a$$

And

$$s(P)=\sum_{j=1}^{n}0(x_j-x_{j-1})=0$$

Since all upper sums equal *b-a* and all lower sums equal 0, Definition 2.6.1 implies that

$$\overline{\int_a^b} f(x)dx=b-a \quad \text{and} \quad \underline{\int_a^b} f(x)dx=0$$

2.7 Riemann Integral

Riemann integral, defined in 1854, was the first of the modern theories of integration and enjoys many of the desirable properties of an integration theory. The groundwork for the Riemann integral of a function f over the interval $[a,b]$ begins with dividing the interval into smaller subintervals.

With infimum and suprimum taken include all partitions P on $[a,b]$, if the upper integral and lower integral same, then f can be said integrable on $[a,b]$.

And called Riemann function f on $[a,b]$ and denoted by $f \in [a,b]$ This same value is called the Riemann integral function f on $f \in [a,b]$ and written

$$(R)\int_a^b f(x)dx$$

(Rahman, 2008:13)

Example 2.7.1:

Let $I := [1,2]$ And $f : A \to R$ with $f(x) = 2x$

We will show $f(x) = 2x$ integrable on $[1,2]$.

Let P_n any partition on $I := [1,2]$ in n subinterval is:

$$P_n = \left(1, 1+\frac{1}{n}, 2+\frac{1}{n}, \cdots, 1+\frac{n-1}{n}, 1+\frac{n}{n} = 2\right)$$

Infimum and supremum from f on subinterval $\left[1+\frac{i-1}{n}, 1+\frac{i}{n}\right]$ is:

$$m_i = 2\left(1+\frac{i-1}{n}\right) \quad and \quad M_i = 2\left(1+\frac{i}{n}\right)$$

also $x_i - x_{i-1} = \left(\frac{1}{n}\right)$ $for \ \ i = 1,2,3,\cdots,n$ then we get:

$$s(P_n) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$
$$= \sum_{i=1}^{n} \left(1 + \frac{i-1}{n}\right)\left(\frac{1}{n}\right)$$
$$= \sum_{i=1}^{n} \left(\frac{2}{n} + \frac{2}{n^2} i - \frac{2}{n^2}\right)$$
$$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{2}{n^2}\sum_{i=1}^{n} 1$$
$$= \frac{2}{n} n + \frac{2}{n^2}\frac{n(n+1)}{2} - \frac{2}{n^2} n$$
$$= 2 + 1 + \frac{1}{n} - \frac{2}{n} = 3 - \frac{1}{n}$$

And we get;

$$S(P_n) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$
$$= \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right)$$
$$= \sum_{i=1}^{n} \left(\frac{2}{n} + \frac{2}{n^2} i\right)$$
$$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{2}{n^2}\sum_{i=1}^{n} i$$
$$= \frac{2}{n} n + \frac{2}{n^2}\frac{n(n+1)}{2}$$
$$= 2 + 1 + \frac{1}{n} = 3 + \frac{1}{n}$$

Since set of partition $\{P_n | n \in N\} \in \wp(I)$, so that

$$3 = \sup\{s(P_n) | n \in N\} \leq \sup\{s(P_n) P \in \wp(I)\} = s(f)$$
and
$$S(f) = \inf\{S(P_n) P \in \wp(I)\} \leq \inf\{S(P_n) | n \in N\} = 3$$

Hence f integrable on $I := [1,2]$ and $\int_1^2 f = \int_1^2 2x dx = 3$.

Definition 2.7.2 (Douglas, 2005:11)

Let $[a,b] \subset \Re$. A partition of $[a,b]$ is a finite set of numbers $P = \{x_0, x_1, ..., x_n\}$ such that $x_0 = a, x_n = b$ and $x_{i-1} < x_i$ for $i = 1,2,...,n$. For each subinterval $[x_{i-1}, x_i]$, define its length to be $\ell([x_{i-1}, x_i]) = x_i - x_{i-1}$. The mesh of the partition is then the length of the largest subinterval, $[x_{-1i}, x_i]$:

$$\mu(P) = \max\{x_i - x_{i-1} : i = 1,2,..., n\}$$

Thus the point $\{x_0, x_1, ..., x_n\}$ form an increasing sequence of numbers in $[a,b]$ that divides the interval $[a,b]$ into contiguous subintervals.

Let $f : [a,b] \to \Re$, $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a,b]$, and $t_i \in [x_{i-1}, x_i]$ for each *i*. Riemann began by considering the approximating (Riemann) sums

$$S(f, P, \{t_i\}_{i=1}^n) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}),$$

Defined with respect to the partition P and the set of sampling points $\{t_i\}_{i=1}^n$. Riemann considered the integral of f over $[a,b]$ to be a "limit" of the sums $S(f, P, \{t_i\}_{i=1}^n)$, in the following sense.

Definition 2.7.3 (Douglas, 2005:12)

A function $f : [a,b] \to \Re$ is Riemann integrable over $[a,b]$ if there is an $A \in \Re$ such that for all $\varepsilon > 0$ there is a $\delta > 0$ so that if P is any partition of $[a,b]$ with $\mu(P) < \delta$ and $t_i \in [x_{i-1}, x_i]$ for all *i* then

$$\left|S(f, P, \{t_i\}_{i=1}^n) - A\right| < \varepsilon$$

We write $A = \int_a^b f = \int_a^b f(t)dt$ or, if we set $I = [a,b]$, $\int_I f$.

This definition defines the integral as a limit of sums as the mesh of the partition approaches 0.

Proposition 2.7.4 (Douglas, 2004: 13)

If *f* is Riemann integrable over $[a,b]$, then the value of the integral is unique.

Proof:

Suppose that f is Riemann integrable over $[a,b]$ and both A and B satisfy Definition 2.7.2. Fix $\varepsilon > 0$ and choose δ_A and δ_B corresponding to A and *B*, respectively, in the definition with $\varepsilon' = \frac{\varepsilon}{2}$. Let $\delta = \min(\delta_A, \delta_B)$ and suppose that *P* is a partition with $\mu(P) < \delta$, and hence with mesh less than both δ_A and δ_B. Let $\{t_i\}_{i=1}^n$ be any set of sampling points for *P*. Then

$$|A - B| \le \left|A - S\left(f, P, \{t_i\}_{i=1}^n\right)\right| + \left|S\left(f, P, \{t_i\}_{i=1}^n\right) - B\right| < \varepsilon' + \varepsilon' = \varepsilon$$

Since ε was arbitrary, it follows that *A*=*B*. Thus, the value of the integral is unique.

Example 2.7.5

Let $a, b, c, d \in \Re$ with $a \le c \le d \le b$. Set $I = [c,d]$ and let χ_I be the characteristic function on I, Define by

$$\chi_I(x) = \begin{cases} 1 & if\ x \in I \\ 0 & if\ x \notin I \end{cases}$$

Then ,

$$\int_a^b \chi_I = d - c$$

Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a,b]$. Let $[x_{i-1}, x_i]$ be a subinterval determined by the partition. The contribution to the Riemann sum from $[x_{i-1}, x_i]$ is either $x_i - x_{i-1}$ or 0 depending on whether or not the sampling point is in *I*.

Now, fix $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{2}$ and let *P* be a partition of $[a,b]$ with mesh less than δ. Let *j* be the smallest index such that $c \in [x_{j-1}, x_j]$ and *k* be the largest index such that $d \in [x_{k-1}, x_k]$. (If $c \in P \setminus \{a,b\}$, then *c* is in two subintervals determined by *P*. Then, if $t_i \in [x_{i-1}, x_i]$ for each *I*,

$$\begin{aligned} S\left(f, P, \{t_i\}_{i=1}^{n}\right) &= f(t_j)(x_j - x_{j-1}) \\ &\quad + \sum_{i=j+1}^{k-1} (x_i - x_{i-1}) + f(t_k)(x_k - x_{k-1}) \\ &< \delta + (d-c) + \delta. \end{aligned}$$

On the other hand

$$\begin{aligned} S\left(f, P, \{t_i\}_{i=1}^{n}\right) &\geq \sum_{i=j+1}^{k-1} (x_i - x_{i-1}) \\ &= \sum_{i=j}^{k} (x_i - x_{i-1}) - \{(x_j - x_{j-1}) + (x_k - x_{k-1})\} \\ &> (d-c) - 2\delta \end{aligned}$$

So that

$$\left| S\left(f, P, \{t_i\}_{i=1}^{n}\right) - (d-c) \right| < 2\delta = \varepsilon$$

Thus, χ_I is Riemann integrable and $\int_a^b \chi_i = d - c$

Example 2.7.6

Define $f:[0,1]\to\Re$ by $f(x)=x$. Let $P=\{x_0,x_1,...,x_n\}$ be a partition of $[0,1]$ and choose t_i so that $x_{i-1}\le t_i\le x_i$. Write $\frac{1}{2}$ as a telescoping sum

$$\frac{1}{2}=\frac{1}{2}\left(x_n^2-x_0^2\right)=\frac{1}{2}\left\{\left(x_1^2-x_0^2\right)+\left(x_2^2-x_0^2\right)+...+\left(x_n^2-x_{n-1}^2\right)\right\}.$$

Then

$$\left|S\left(f,P,\{t_i\}_{i=1}^n\right)-\frac{1}{2}\right|=\left|\sum_{i=1}^n t_i\left(x_i-x_{i-1}\right)-\frac{1}{2}\sum_{i=1}^n t_i\left(x_i^2-x_{i-1}^2\right)\right|$$
$$=\left|\sum_{i=1}^n\left(t_i-\frac{x_i-x_{i-1}}{2}\right)\left(x_i-x_{i-1}\right)\right|$$

Since $t_i,\frac{x_i+x_{i-1}}{2}\in[x_{i-1},x_i],\left|t_i-\frac{x_i+x_{i-1}}{2}\right|\le\mu(P)$. so, given $\varepsilon>0$, set $\delta=\varepsilon$.

Then , if $\mu(P)<\delta$,

$$\left|S\left(f,P,\{t_i\}_{i=1}^n\right)-\frac{1}{2}\right|\le\sum_{i=1}^n\left|\left(t_i-\frac{x_i-x_{i-1}}{2}\right)\left(x_i-x_{i-1}\right)\right|$$
$$<\delta\sum_{i=1}^n\left(x_i-x_{i-1}\right)=\delta=\varepsilon$$

Since $\sum_{i=1}^n\left(x_i-x_{i-1}\right)=1$. Thus, *f* is Riemann integral on $[0,1]$ and has integral $\frac{1}{2}$.

Example 2.7.7

Define the Dirichlet function $f:[0,1]\to\Re$ by

$$f(x)=\begin{cases}1 & if\ x\in Q\\ 0 & if\ x\notin Q\end{cases}$$

Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[0,1]$. In every subinterval $[x_{i-1}, x_i]$ there is rational number r_i and an irrational number q_i. Thus,

$$S\left(f, P, \{r_i\}_{i=1}^{n}\right) = \sum_{i=1}^{n} f(r_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} 0 = 0$$

While

$$S\left(f, P, \{q_i\}_{i=1}^{n}\right) = \sum_{i=1}^{n} f(q_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1}) = 1$$

So, no matter how fine the partition, we can always find a set of sampling points so that the corresponding Riemann sum equals 1. Now, suppose f were Riemann integrable with integral A. fix $\varepsilon < \frac{1}{2}$ and choose a corresponding δ. If P is any partition with mesh less than δ, then

$$\begin{aligned} 1 &= \left|S\left(f, P, \{q_i\}_{i=1}^{n}\right) - S\left(f, P, \{r_i\}_{i=1}^{n}\right)\right| \\ &\le \left|S\left(f, P, \{q_i\}_{i=1}^{n}\right) - A\right| + \left|A - S\left(f, P, \{r_i\}_{i=1}^{n}\right)\right| < \varepsilon + \varepsilon < 1 \end{aligned}$$

This contradiction shows that f is not Riemann integrable.

2.8 Obligations Studying Mathematics

Allah SWT has ordered His servants to make Al-Quran and the sunnah as the main source of knowledge. This is because both are directly from Allah SWT and in its supervision, so the wake of the mistake. In QS Luqman verse 27 Allah says:

وَلَوْ أَنَّمَا فِى ٱلْأَرْضِ مِن شَجَرَةٍ أَقْلَـٰمٌ وَٱلْبَحْرُ يَمُدُّهُۥ مِنۢ بَعْدِهِۦ سَبْعَةُ أَبْحُرٍ مَّا نَفِدَتْ كَلِمَـٰتُ ٱللَّهِ ۗ إِنَّ ٱللَّهَ عَزِيزٌ حَكِيمٌ ﴿٢٧﴾

"and if all the trees on earth were pens and the sea (the ink), added to it seven sea (again) after (dry) it, would not inexhaustible (written) sentence of God. Surely Allah is Mighty, Wise" (QS Luqman: 27)

The meaning of sentence of God is the His science and His blessings because so vast knowledge of Allah and so abundant blessings derived to human. So that if the sea became ink for the writing of science and the bounty of Allah SWT to humans. Even added seven times would not be enough to write the science and the blessings of Allah. As mentioned in the QS Al-Kahfi verse 109:

قُل لَّوْ كَانَ ٱلْبَحْرُ مِدَادًا لِّكَلِمَـٰتِ رَبِّى لَنَفِدَ ٱلْبَحْرُ قَبْلَ أَن تَنفَدَ كَلِمَـٰتُ رَبِّى وَلَوْ جِئْنَا بِمِثْلِهِۦ مَدَدًا ﴿١٠٩﴾

"Say: Had oceans become ink for (write) sentences sovereign, really screwed sea would before discharged (written) sentences sovereign, although we bring additional much (too)". (QS Al-Kahfi : 109)

Analysis is a method used to investigate and explain a case. With further analysis, people can think more logically and more precise about everything, especially about the blessing of God in the form sense, so that with the reason and mind, human could get knowledge more, because knowledgeable people have a higher degree in the eyes of Allah SWT from the people who are not knowledgeable.

CHAPTER III
DISCUSSION

3.1 Concept δ-fine partition of Henstock-Kurzweil Integral

Let $[a,b]$ be a compact interval in $\Re$. Let D be a finite collection of interval-point pairs $\{([u_i,v_i],\xi_i)\}_{i=1}^{n}$, where $\{([u_i,v_i])\}_{i=1}^{n}$ are non-overlapping subintervals of $[a,b]$. Let $\delta(\xi)$ be a positive function on $[a,b]$, $i.e.\delta(\xi):[a,b]\to\Re^{+}$. We say $D=\{([u_i,v_i],\xi_i)\}_{i=1}^{n}$ is δ-fine Henstock-Kurzweil partition of $[a,b]$ if $\xi_i\in[u_i,v_i]\subset B(\xi_i,\delta(\xi_i))=(\xi_i-\delta(\xi_i),\xi_i+\delta(\xi_i))$ for all $i=1,2,3,...,n$

Given an δ-fine Henstock-Kurzweil partition $D=\{([u_i,v_i],\xi_i)\}_{i=1}^{n}$ we write

$$S(f,D)=\sum_{i=1}^{n} f(\xi_i)(v_i-u_i)$$

For integral sum over D, whenever $f:[a,b]\to\Re$

Example 3.1.1

Let $f(x)=x$ Consider a division $a=x_0<x_1<...<x_n=b$ *and* $\{\xi_1,\xi_2,...,\xi_n\}$. And this time choose the points $\xi_i=\frac{1}{2}(x_i+x_{i-1})$, Clearly $\xi_i\in[x_{i-1},x_i]$ For $i=1,2,\cdots,n$

Then,

$$S(f,D)=\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^{n}\frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1})$$
$$=\frac{1}{2}\sum_{i=1}^{n}\left(x_i^2 - x_{i-1}^2\right)$$
$$=\frac{1}{2}\left(x_n^2 - x_0^n\right)$$
$$=\frac{1}{2}\left(b^2 - a^2\right)$$

Then,

$$S(f,D)=\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})=\frac{1}{2}\left(b^2 - a^2\right)$$

3.2 Definition of Henstock-Kurzweil Integral

Definition 3.2.1

A function $f:[a,b]\rightarrow\Re$ is said to be Henstock-Kurzweil integrable on $[a,b]$ if there exists a real number A such that for every $\varepsilon > 0$ there exists $\delta:[a,b]\rightarrow\Re^{+}$ such that for every δ-fine Henstock-Kurzweil partition $D=\{([u_i,v_i],\xi_i)\}_{i=1}^{n}$ of $[a,b]$, we have

$$\left|\sum_{i=1}^{n} f(\xi_i)(v_i - u_i) - A\right| < \varepsilon$$

We denote the Henstock-Kurzweil Integral (also write as HK-integral) A by $(HK)\int_a^b f(x)dx$.

Example 3.2.2

Define $f:[0,1]\to\Re$ the Dirichlet's function (= the characteristic function of the rational numbers in $[0,1]$), by

$$f(x)=\begin{cases}1 & if\ x\in Q\\ 0 & if\ x\notin Q\end{cases}$$

Then $f(x)$ is Henstock-Kurzweil integrable on $[0,1]$. And $\int_0^1 f(x)=0$

To Prove this assertion, we will define an appropriate gauge δ_ε, First we enumerate these rational numbers as $r_1, r_2, \dots$. We define $\delta(r_i)=\varepsilon 2^{-i-1}\ for\ i=1,2,\dots$, and if $x\in[0,1]$ is irrational we define $\delta(\xi)=1$; clearly δ_ε is a gauge on $[0,1]$. If P is a δ-fine tagged partition, there can be at most two subintervals in P that have the number r_i as tag, and the length of each of those subintervals is $\le\varepsilon 2^{-i-1}$. Hence the contribution to $S(f,P)$ from subintervals with tag r_i is $\le\varepsilon 2^{-i}$. Since the terms in $S(f,P)$ with tags at irrational points contribute 0, we readily see that

$$0\le S(f,P)<\sum_{i=1}^{\infty}\varepsilon/2^i=\varepsilon$$

Since $\varepsilon>0$ is arbitrary, this shows that $f(x)$ is Henstock-Kurzweil integrable on $[0,1]$. And $\int_0^1 f(x)=0$

Example 3.2.3

Let $f(x) = x^{-2}$ when $1 \le x < +\infty$. Given $\varepsilon > 0$, define $\delta(x) = \varepsilon x$ when $1 \le x < +\infty$ and $\delta(+\infty) = 1/\varepsilon$ then for any $\delta - fine$ division as given above in the definition with $x_0 = 1 x_n > 1/\varepsilon$ we have

$$\left|\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - 1\right| \le \left|\sum_{i=1}^{n} \frac{x_i - x_{i-1}}{\xi_i^2} - \sum_{i=1}^{n} \frac{1}{x_{i-1}} - \frac{1}{x_i}\right| + \frac{1}{x_n}$$
$$\le \sum_{i=1}^{n} \left|\frac{x_i - x_{i-1}}{\xi_i^2} - 1\right| \left(\frac{1}{x_{i-1}} - \frac{1}{x_i}\right) + \varepsilon$$

Considering separately the case when $x_i x_{i-1}/\xi_i^2 - 1$ is positive or negative, we obtain

$$\left|\frac{x_i - x_{i-1}}{\xi_i^2} - 1\right| \le \frac{x_i - x_{i-1}}{\xi_i} < 2\varepsilon$$

Combining the above inequalities, we show that *f* is Henstock-Kurzweil integrable on $[1, +\infty)$.

3.3 Fundamental Properties Of Henstock-Kurzweil Integral

Theorem 3.3.1 (Unique Property)

if *f* is Henstock-Kurzweil integrable over $[a, b]$, then the value of the integral is unique.

Proof:

Suppose that *f* is Henstock-Kurzweil integrable on $[a, b]$ and both real number *A* and *B* satisfy Definition 3.2.1. *Fix* $\varepsilon > 0$ choose δ_A and δ_B corresponding to *A* and *B*, respectively, in the definition with $\varepsilon' = \frac{\varepsilon}{2}$. Let $\delta = \min(\delta_A, \delta_B)$ and suppose for every δ-fine Henstock-Kurzweil partition $D=\{([u_i,v_i],\xi_i)\}_{i=1}^{n}$ *of* $[a,b]$

, Then

$$\begin{aligned}|A-B| &= \left|\left(\sum_{i=1}^{n} f(\xi_i)(v_i-u_i)-B\right)-\left(\sum_{i=1}^{n} f(\xi_i)(v_i-u_i)-A\right)\right| \\ &\leq \left|A-\sum_{i=1}^{n} f(\xi_i)(v_i-u_i)\right|+\left|\sum_{i=1}^{n} f(\xi_i)(v_i-u_i)-B\right| \\ &< \varepsilon'+\varepsilon'=\varepsilon\end{aligned}$$

Since ε was arbitrary, it follows that $A=B$. Thus, the value of the integral is unique.

Theorem 3.3.2 (Linearity of the Henstock-Kurzweil integral)

If f and g are Henstock-Kurzweil integrable on $[a,b]$, then so are $f+g$ and αf where α is real. Furthermore,

$$\int_a^b (f+g)=\int_a^b f+\int_a^b g \text{ and } \int_a^b (\alpha f)=\alpha\int_a^b f$$

Proof: Let A and B denote respectively the integrals of f and g on $[a,b]$. given $\varepsilon>0$, there is a $\delta_1(\xi)>0$ such that for any δ_1-fine division $D=([u,v];\xi)$ we have

$$\left|\sum f(\xi)(v-u)-A\right|<\frac{\varepsilon}{2}$$

Similarly, there is a $\delta_2(\xi)>0$ such that for any δ_2-fine division $D=([u,v];\xi)$ we have

$$\left|\sum g(\xi)(v-u)-B\right|<\frac{\varepsilon}{2}$$

Now put $\delta(\xi)=\min(\delta_1(\xi),\delta_2(\xi))$. Note that any δ-fine division is also δ_1-fine and δ_2-fine. Therefore for any δ-fine division $D=([u,v];\xi)$ we have

$$\left|\sum(f(\xi)+g(\xi))(v-u)-(A+B)\right| \leq \left|\sum f(\xi)(v-u)-A\right|+\left|\sum g(\xi)(v-u)-B\right|$$

$$< \varepsilon$$

The proof is complete.

Example 3.3.3

Evaluate $\int_0^1 f+g$ where $f(x)=x^2$ and $g(x)=x$.

Solution:

By theorem 3.3.3,

$$\int_0^1 f+g=\int_0^1 x^2+x\,dx=\int_0^1 x^2\,dx+\int_0^1 x\,dx$$

Now, we evaluate value $\int_0^1 x^2dx$

Consider a division $0=x_0<x_1<\ldots<x_n=1$ *and* $\{\xi_1,\xi_2,\ldots,\xi_n\}$. And this time choose the intermediate points

$$\xi_i=\left[\frac{1}{3}\left(x_i^2+x_ix_{i-1}+x_{i-1}^2\right)\right]^{1/2}, \text{ then}$$

$$0\leq x_{i-1}=\left(x_{i-1}^2\right)^{1/2}<\left[\frac{1}{3}\left(x_i^2+x_ix_{i-1}+x_{i-1}^2\right)\right]^{1/2}<\left(x_i^2\right)^{1/2}=x_i$$

For $i=1,2,\cdots,n$; that is $\xi_i\in(x_{i-1},x_i)$ for each i.

So $\int_0^1 x^2dx=\frac{1}{3}$

And then , we evaluate value $\int_0^1 xdx$

Consider a division $0 = x_0 < x_1 < ... < x_n = 1$ *and* $\{\xi_1, \xi_2, ..., \xi_n\}$. And this time choose the points $\xi_i = \frac{1}{2}(x_i + x_{i-1})$, Clearly $\xi_i \in [x_{i-1}, x_i]$ For $i = 1, 2, \cdots, n$

Now

$$S(f, D) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1})$$
$$= \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2)$$
$$= \frac{1}{2}(x_n^2 - x_0^n)$$
$$= \frac{1}{2} \cdot 1^2$$
$$= \frac{1}{2} \cdot 1 \quad = \frac{1}{2}$$

So, $\int_0^1 x dx = \frac{1}{2}$

So that,

$$\int_0^1 f + g = \int_0^1 x^2 + x\, dx = \int_0^1 x^2\, dx + \int_0^1 x\, dx = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

Theorem 3.3.4 (Additivity of the henstock-Kurzweil Integral)

Let $a < c < b$. If f is Henstock-Kurzweil integrable on $[a, c]$ and on $[c, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof: Let A denote the integral of f on $[a, c]$ and B that of f on $[c, b]$. Given $\varepsilon > 0$, there is a $\delta_1(\xi) > 0$, defined on $[a, c]$, such that for any δ_1-fine division $D = ([u, v]; \xi)$ of $[a, c]$ we have

$$\left|\sum f(\xi)(v-u)-A\right|<\frac{\varepsilon}{2}$$

Similarly, there is a $\delta_2(\xi)>0$ defined on $[c,b]$ such that for any δ_2-fine division $D=([u,v];\xi)$ of $[c,b]$ we have

$$\left|\sum f(\xi)(v-u)-B\right|<\frac{\varepsilon}{2}$$

Define $\delta(\xi)=\min(\delta_1(\xi),c-\xi)$ when $\xi\in[a,c)$, $\min(\delta_2(\xi),\xi-c)$ when $\xi\in(c,b]$, and $\min(\delta_1(c),\delta_2(c))$ when $\xi-c$. Note for any δ-fine division D of $[a,b]$, c is always a division point of D. therefore for any δ-fine division $D=([u,v];\xi)$ of $[a,b]$ with Σ over D, writing $\Sigma=\Sigma_1+\Sigma_2$ where Σ_1 is the partial sum over $[a,c]$ and Σ_2 over $[c,b]$ we have

$$\left|\sum f(\xi)(v-u)-(A+B)\right|\leq\left|\sum{}_1 f(\xi)(v-u)-A\right|+\left|\sum{}_2 f(\xi)(v-u)-B\right|$$

$$<\varepsilon$$

Hence *f* is Henstock-Kurzweil integrable to A+B on $[a,b]$.

Alternatively, Let $\chi_{[a,c]}$ denote the characteristic function of $[a,c]$ and $f_1=f\chi_{[a,c]}$. Similarly, let $f_2=f\chi_{[c,b]}$. Then it follows from Theorem 3.3.1 that

$$\int_a^b f=\int_a^b(f_1+f_2)=\int_a^c f+\int_c^b f$$

Example 3.3.5

Let $f(x) = x$ and Let $0 < \frac{1}{2} < 1$.and, If f is Henstock-Kurzweil integrable on $[a,c] = \left[0, \frac{1}{2}\right]$ and on $[c,b] = \left[\frac{1}{2}, 1\right]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Solution:

Consider a division $0 = x_0 < x_1 < ... < x_n = 1$ and $\{\xi_1, \xi_2, ..., \xi_n\}$. And this time choose the points $\xi_i = \frac{1}{2}(x_i + x_{i-1})$, Clearly $\xi_i \in [x_{i-1}, x_i]$ For $i = 1, 2, \cdots, n$

Now

$$\begin{aligned} S(f, D) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) &= \sum_{i=1}^{n} \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \frac{1}{2}\sum_{i=1}^{n}\left(x_i^2 - x_{i-1}^2\right) \\ &= \frac{1}{2}\left(x_n^2 - x_0^2\right) \\ &= \frac{1}{2}\left(b^2 - a^2\right) \end{aligned}$$

With same procedure we get; on $[a,c]=\left[0,\frac{1}{2}\right]$

$$S(f,D)=\sum_{i=1}^{n} f(\xi_i)(x_i-x_{i-1})=\sum_{i=1}^{n}\frac{1}{2}(x_i+x_{i-1})(x_i-x_{i-1})$$
$$=\frac{1}{2}\sum_{i=1}^{n}(x_i^2-x_{i-1}^2)$$
$$=\frac{1}{2}(x_n^2-x_0^2)$$
$$=\frac{1}{2}(c^2-a^2)$$

So $\varepsilon>0$, there is a $\delta_1(\xi)>0$, defined on $\left[0,\frac{1}{2}\right]$, such that for any δ_1-fine division $D=([u,v];\xi)$ of $\left[0,\frac{1}{2}\right]$ we have

$$\left|\frac{1}{2}(c^2-a^2)-\frac{1}{8}\right|<\frac{\varepsilon}{2}$$

And on $[c,b]=\left[\frac{1}{2},1\right]$, we get

$$S(f,D)=\sum_{i=1}^{n} f(\xi_i)(x_i-x_{i-1})=\sum_{i=1}^{n}\frac{1}{2}(x_i+x_{i-1})(x_i-x_{i-1})$$
$$=\frac{1}{2}\sum_{i=1}^{n}(x_i^2-x_{i-1}^2)$$
$$=\frac{1}{2}(x_n^2-x_0^2)$$
$$=\frac{1}{2}(b^2-c^2)$$

$\varepsilon > 0$, there is a $\delta_2(\xi) > 0$, defined on $\left[\frac{1}{2}, 1\right]$, such that for any δ_1-fine division $D = ([u,v];\xi)$ of $\left[\frac{1}{2}, 1\right]$ we have

$$\left|\frac{1}{2}(b^2 - c^2) - \frac{3}{8}\right| < \frac{\varepsilon}{2}$$

therefore for any δ-fine division $D = ([u,v];\xi)$ of $[0,1]$ with Σ over D, writing $\Sigma = \Sigma_1 + \Sigma_2$ where Σ_1 is the partial sum over $[a,c]$ and Σ_2 over $\left[\frac{1}{2}, 1\right]$ we have

$$\left|\frac{1}{2}(b^2 - a_2) - \left(\frac{1}{8} + \frac{3}{8}\right)\right| \leq \left|\sum{}_1 f(\xi)(v-u) - \frac{1}{8}\right| + \left|\sum{}_2 f(\xi)(v-u) - \frac{3}{8}\right|$$

$$< \varepsilon$$

Hence *f* is Henstock-Kurzweil integrable to $\frac{1}{2}$ on $[0,1]$

Lemma 3.3.6 (Cauchy Criteria)

A function is Henstock-kurzweil integrable on $[a,b]$ if and only if for every $\varepsilon > 0$, there is a $\delta(\xi) > 0$ such that for any δ-fine division $D = ([u,v];\xi)$ and $D' = ([u',v'];\xi')$ we have

$$\left|\sum f(\xi)(v-u) - \sum f(\xi')(v'-u')\right| < \varepsilon$$

Where the first sum is over *D* and the second over *D*'.

Proof

$(\Rightarrow)$ we will prove that if A function is Henstock-kurzweil integrable on $[a,b]$ Then for every $\varepsilon > 0$, there is a $\delta(\xi) > 0$ such that for any δ-fine division $D = ([u,v];\xi)$ and $D' = ([u',v'];\xi')$ we have

$$\left|\sum f(\xi)(v-u)-\sum f(\xi')(v'-u')\right|<\varepsilon$$

A function is Henstock-kurzweil integrable on $[a,b]$ Then for every $\varepsilon>0$, there is a $\delta(\xi)>0$ such that for any δ-fine division $D=([u,v];\xi)$ and $D'=([u',v'];\xi')$

$$\begin{aligned}\left|\sum f(\xi)(v-u)-\sum f(\xi')(v'-u')\right| &= \left|\sum f(\xi)(v-u)-A+A-\sum f(\xi')(v'-u')\right|\\ &\le \left|\sum f(\xi)(v-u)-A\right|+\left|A-\sum f(\xi)(v-u)\right|\\ &<\varepsilon\end{aligned}$$

Analogous to the situation for real-valued sequences, the condition that

$$\left|\sum f(\xi)(v-u)-\sum f(\xi')(v'-u')\right|<\varepsilon$$

$(\Leftarrow)$ We have already proved that the integrability of f implies the Cauchy criterion. So, assume the Cauchy criterion holds. We will prove that *f* is Henstock-kurzweil integrable .

if for every $\varepsilon>0$, there is a $\delta(\xi)>0$ such that for any δ-fine division $D=([u,v];\xi)$ and $D'=([u',v'];\xi')$ we have

$$\left|\sum f(\xi)(v-u)-\sum f(\xi')(v'-u')\right|<\varepsilon$$

Then A function is Henstock-Kurzweil integrable on [a,b].

For each $k\in \mathrm{N}$, choose a $\delta_k>0$ so that for any two division $D=([u,v];\xi)$ *and* $D'=([u',v'];\xi')$, and corresponding sampling points, we have

$$\left|\sum f(\xi)(v-u)-\sum f(\xi')(v'-u')\right|<\frac{1}{k}$$

Replacing δ_k by $\min\{\delta_1,\delta_2,...,\delta_k\}$, we may assume that $\delta_k\ge\delta_{k+1}$.

Next for each *k*, fix a partition $D_k = ([u_k, v_k]; \xi_k)$ and set of sampling point $\{\xi_i\}_{i=1}^n$. Note for $j > k$ Thus

$$\left|\sum f(\xi_k)(v_k - u_k) - \sum f(\xi_j)(v_j - u_j)\right| < \frac{1}{\min\{j,k\}},$$

Which implies that sequence $\sum_{k=1}^{\infty} f(\xi_k)(v_k - u_k)$ is a Cauchy sequence in *R*, and hence converges. Let *A* be a limit of this sequence. it follows from the previous inequality that

$$\left|\sum f(\xi)(v-u) - A\right| < \frac{1}{k}$$

It remains to show that *A* satisfies Definition 3.2.1

Fix $\varepsilon > 0$ and let division $D = ([u,v];\xi)$. Then

$$\left|\sum f(\xi)(v-u) - A\right|$$
$$= \left|\sum_{i=1}^{n} f(\xi_i)(v_i - u_i) - \sum_{k=1}^{n} f(\xi_k)(v_k - u_k) + \sum_{k=1}^{n} f(\xi_k)(v_k - u_k) - A\right|$$
$$\leq \left|\sum_{i=1}^{n} f(\xi_i)(v_i - u_i) - \sum_{k=1}^{n} f(\xi_k)(v_k - u_k)\right| + \left|\sum_{k=1}^{n} f(\xi_k)(v_k - u_k) - A\right|$$
$$< \frac{1}{k} + \frac{1}{k} = \varepsilon$$

It now follows that *f* is Henstock-Kurzweil integrable on $[a,b]$

Theorem 3.3.8

If *f* is Henstock-Kurzweil integrable on $[a,b]$, then so it is on a subinterval $[c,d]$ of $[a,b]$.

Proof :

Since f is Henstock-Kurzweil integrable on $[a,b]$, the Cauchy condition holds. Take any two δ-fine divisions of $[c,d]$. say D_1 and D_2, and denote by s_1 and s_2 respectively the Riemann sums of f over D_1 and D_2. similarly, take another δ-fine division D_3 of $[a,c]\cup[d,b]$ and denote by s_3 the corresponding Riemann sums. Then the union $D_1 \cup D_3$ forms a δ-fine division of $[a,b]$. Here the division points and associated points of $D_1 \cup D_3$ are the union of those from D_1 and D_3. The Riemann sum of f over $D_1 \cup D_3$ is $s_1 + s_3$. And similarly that over $D_2 \cup D_3$ is $s_2 + s_3$.Therefore by the Cauchy condition we have

$$|s_1 - s_2| \le |(s_1 + s_3) - (s_2 + s_3)| < \varepsilon$$

Hence the result follows from lemma 3.3.7 with $[a,b]$replaced by $[c,d]$

Theorem 3.3.9 (Nonnegativity of The Henstock-Kurzweil integral)

If f and g are Henstock-Kurzweil integrable on $[a,b]$and if $f(x) \le g(x)$ for almost all in x in $[a,b]$, then

$$\int_a^b f \le \int_a^b g$$

Proof:

In view of theorem 3.3.9 we may assume that $f(x) \le g(x)$ for all x. Given $\varepsilon > 0$, as in the proof of theorem 3.2.1, there is a $\delta(\xi) > 0$ such that for any δ-fine division $D = ([u,v];\xi)$ we have

$$\left|\sum f(\xi)(v-u)-\int_a^b f\right|<\varepsilon\,,$$

$$\left|\sum g(\xi)(v-u)-\int_a^b g\right|<\varepsilon$$

It follows that

$$\int_a^b f-\varepsilon<\sum f(\xi)(v-u)<\sum g(\xi)(v-u)<\int_a^b g+\varepsilon$$

Since ε is arbitrary, we have the required inequality.

Theorem 3.3.11

If *f* is Henstock-Kurzweil integrable on $[a,b]$ with the primitive *F*, then for every $\varepsilon>0$, there is a $\delta(\xi)>0$ such that for any δ-fine division $D=([u,v];\xi)$ we have

$$\sum\left|F(v)-F(u)-f(\xi)(v-u)\right|<\varepsilon$$

We shall make a few remarks. Before proof, from the computational point of view, we may regard $f(\xi)(v-u)$ as an approximation of $F(v)-F(u)$. Then the difference $F(v)-F(u)-f(\xi)(v-u)$ is an error. The definition of the Henstock-Kurzweil integral says that the absolute error is also small, whereas Henstock'Lemma. In fact, the two are equivalent by theorem 3.3.8. Another way of putting it is that taking any partial sum Σ_1 of Σ we still have

$$\left|\Sigma_1 F(v)-F(u)-f(\xi)(v-u)\right|<\varepsilon$$

That is to say, the selected error is again small, and indeed it is equivalent to the above two.

Proof:

Given $\varepsilon > 0$, there is a $\delta(\xi) > 0$ such that for any δ-fine division $D = ([u,v];\xi)$ we have

$$\left|\Sigma F(v) - F(u) - f(\xi)(v-u)\right| < \varepsilon/4$$

Let Σ_1 be a partial sum of Σ and E_1 the union of $[u,v]$ from Σ_1. Suppose E_2. thus we can choose a δ-fine division $D_2 = ([u,v];\xi)$ of E_2 such that

$\left|\Sigma_2 F(v) - F(u) - f(\xi)(v-u)\right| < \varepsilon/4$

Where Σ_2 is over D_2. Now writing $\Sigma_3 = \Sigma_1 + \Sigma_2$ we have

$$\left|\Sigma_1(F(v) - F(u) - f(\xi)(v-u))\right| \le \left|\Sigma_3(F(v) - F(u) - f(\xi)(v-u))\right| + \left|\Sigma_2(F(v) - F(u) - f(\xi)(v-u))\right|$$

$$< \varepsilon/2$$

Consequently the result follows.

Example 3.3.12

Let $f(x) = 1/\sqrt{x}$ for $0 < x \le 1$. Given $\varepsilon > 0$, we shall construct $\delta(\xi)$ so that *f* is Henstock- Kurzweil integrable on $[0,1]$. Consider a division

$0 = x_0 < x_1 < ... < x_n = 1$ and $\{\xi_1, \xi_2, ..., \xi_n\}$

With $\xi_1 = 0$ and $x_{i-1} \le \xi_i \le x_i$ for $i = 2,...,n$. Note that the primitive of $1/\sqrt{x}$ is $2\sqrt{x}$. Then we can write

$$\left|2-\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})\right| \le \left|2-\left(2-\sqrt{x_1}\right)\right|$$

$$+\left|\int_{x_1}^{1} dx\Big/\sqrt{x} - \sum_{i=2}^{n}(x_i - x_{i-1})\Big/\sqrt{\xi_i}\right|$$

$$\le 2\sqrt{x_1} + \sum_{i=2}^{n}\left(1\Big/\sqrt{x_{i-1}} - 1\Big/\sqrt{x_i}\right)(x_i - x_{i-1}).$$

We shall prove that above is less than ε for suitable $\delta-fine$ divisions. Suppose $\delta(\xi)-c\xi$ for $0<\xi\le 1$ and $0<c<1/2$ so that $\xi_1 = 0$ always. If the above division is $\delta-fine$ and $[u,v]$ is a typical interval $[x_{i-1}, x_i]$ in the division with $u \ne 0$ and $u \le \xi \le v$, then $0 < v-u < 2\delta(\xi) \le 2cv$, re-arranging we get $v/u \le 1/(1-2c)$, and finally

$$(v-u)/\sqrt{uv} < 2cv/u \le 2c/(1-2c).$$

Now choose c so that $0<c<1/2$ and $2c/(1-2c) \le \varepsilon/2$. In addition, put $\delta(0) \le \varepsilon^2/16$. Then for the given $\delta-fine$division the above inequality is less than

$$2\sqrt{\delta(0)} + \frac{2c}{1-2c}\sum_{i=2}^{n}\left(\sqrt{x_i} - \sqrt{x_{i-1}}\right) < \varepsilon$$

For example, when $0<\varepsilon\le 1$ we may choose $c = \dfrac{\varepsilon}{6}$. Hence the function is

Henstock-Kurzweil integrable on $[0,1]$.

3.4 Linkage analysis Henstock-Kurzweil Integral with Al-Quran

Everything in the universe there are size, count and formulas or equations. Mathematician or a physicist does not make a formula one bit, but they only found a formula or equation. If in life there is a problem, people should try to find a solution.

Therefore very important for people to learn math because Al-Quran have teach human being about the sum, subtraction, equality and others. By learning math, not only to grow the way of thinking which systematic and consistent, but also expected to grow a thorough manner. In this regard Allah says in QS Al-Mukminun verses 112-114 as follows:

قَٰلَ كَمۡ لَبِثۡتُمۡ فِي ٱلۡأَرۡضِ عَدَدَ سِنِينَ ﴿١١٢﴾ قَالُواْ لَبِثۡنَا يَوۡمًا أَوۡ بَعۡضَ يَوۡمٖ فَسۡـَٔلِ ٱلۡعَآدِّينَ ﴿١١٣﴾
قَٰلَ إِن لَّبِثۡتُمۡ إِلَّا قَلِيلٗاۖ لَّوۡ أَنَّكُمۡ كُنتُمۡ تَعۡلَمُونَ ﴿١١٤﴾

112. God asked: "How many years did you stay on earth?"
113. they replied: "We stayed (on earth) a day or half day, then Ask the people who count."
114. He said: "You do not stay (on earth) but a little, if you know the fact" (QS Al-Mukminun : 112-114)

Mathematics called arithmetic because mathematics is essentially related to the numbers. Learning the numbers get encouragement through the Al-Quran which opens new horizons in the field of mathematics.

In calculating affairs, Allah says in the QS Al-Baqarah verse 202:

أُوْلَٰٓئِكَ لَهُمۡ نَصِيبٞ مِّمَّا كَسَبُواْۚ وَٱللَّهُ سَرِيعُ ٱلۡحِسَابِ ﴿٢٠٢﴾

202. That those people who gets a share than they have earned and Allah is quick in reckoning. (QS Al-Baqarah: 202) and confirmed in QS Ali Imran verse 199:

وَإِنَّ مِنْ أَهْلِ ٱلْكِتَٰبِ لَمَن يُؤْمِنُ بِٱللَّهِ وَمَآ أُنزِلَ إِلَيْكُمْ وَمَآ أُنزِلَ إِلَيْهِمْ خَٰشِعِينَ لِلَّهِ لَا يَشْتَرُونَ
بِـَٔايَٰتِ ٱللَّهِ ثَمَنًا قَلِيلًا ۗ أُو۟لَٰٓئِكَ لَهُمْ أَجْرُهُمْ عِندَ رَبِّهِمْ ۗ إِنَّ ٱللَّهَ سَرِيعُ ٱلْحِسَابِ ﴿١٩٩﴾

199. Indeed among the scribes and there are people who believe in Allah and to what is revealed to you, and they handed down to them being humble to God and they do not redeem the Signs of Allah with a little price. their reward with their Lord. And Allah is swift in reckoning. (QS Ali Imran: 199)

Learning math is a tool for other sciences, and also to draw closer to God, because God is the King of counting. Allah is the Mathematician, the mathematical proof of Allah is so clear in this universe, in giving the reward and the issue of prayer. Qur'an describes the multiplication of numbers and calculations in a variety of events. God is a king of calculation affairs, God is very quick in calculating and meticulous.

As explained in the QS Al-Jinn verse 28, states that God created all things (events and all objects in the universe) with a careful count one by one.

لِّيَعْلَمَ أَن قَدْ أَبْلَغُوا۟ رِسَٰلَٰتِ رَبِّهِمْ وَأَحَاطَ بِمَا لَدَيْهِمْ وَأَحْصَىٰ كُلَّ شَىْءٍ عَدَدًۢا ﴿٢٨﴾

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28. so that He may know that the fact that messengers have delive their Lord treatises, are (actually) His knowledge encompasses what is with them, and He counted all things one by one. (QS Al-Jinn: 28)

From the verse above can be concluded that God is the king of counting, because God Almighty is very fast in calculating and meticulous. This should encourage people to always be careful and cautious in doing a job. Once emphasized accuracy and precision in daily life.

Basically, the integral calculation is the summation of all the area which is determined by the number of partitions. Speaking about Henstock-Kurzweil integral, the operation of computation used are addition and multiplication operations. While the addition operation is also mentioned in QS Al-Kahfi verse 25:

وَلَبِثُواْ فِي كَهۡفِهِمۡ ثَلَٰثَ مِاْئَةٖ سِنِينَ وَٱزۡدَادُواْ تِسۡعٗا ٢٥

"and they stayed in their Cave three hundred years and add nine (more)." (QS Al-Kahfi : 25)

This shows in general, concluded that the Quran is mathematically designed, so it can appreciated not only by those who have high-level mathematics, but also by other people.

One of Fundamental properties Henstock Kurzweil integral is uniqueness. If f is Henstock-Kurzweil integrable over $[a,b]$, then the value of the integral is unique. For this property, we can associate with religious studies. Uniqueness in religious studies, can we found about uniqueness of God.

Uniqueness of God is not a numerical singleness, because two, three and four and so on at all impossible for God. His uniqueness is not found in the particular uniqueness of an individual from a natural thing; not also a generic or specific uniqueness found in every general idea. His uniqueness is also the uniqueness of the conjunctively not encountered when a number of things arranged or assembled into a single thing; not also the uniqueness of the relationship found in the quantities and measurable things.

In QS Al-Ikhlas verse 4:

وَلَمۡ يَكُن لَّهُۥ كُفُوًا أَحَدُۢ ٤

"And no one is equal to him"(QS Al-Ikhlas: 4)

Besides uniqueness of God, there are uniqueness of Islam religion. Allah SWT say on QS Ash Shaff verse 9:

هُوَ ٱلَّذِىٓ أَرۡسَلَ رَسُولَهُۥ بِٱلۡهُدَىٰ وَدِينِ ٱلۡحَقِّ لِيُظۡهِرَهُۥ عَلَى ٱلدِّينِ كُلِّهِۦ وَلَوۡ كَرِهَ ٱلۡمُشۡرِكُونَ ٩
.

"He sent His apostle with guidance and true religion that He may win it over all religions even though the polytheists hate it." (QS Ash Shaff: 9)

From the words we can conclude that religion is divided into two parts, the religion that comes from God, the blessed religion of Islam and religion other than Islam. He also emphasized the verse is the true religion of Islam.

The proof that Islam is the only religion that God blessed, can also be seen in the last revelation received by the Prophet in verse 3 QS Al-Maidah, which reads:

ٱلۡيَوۡمَ أَكۡمَلۡتُ لَكُمۡ دِينَكُمۡ وَأَتۡمَمۡتُ عَلَيۡكُمۡ نِعۡمَتِي وَرَضِيتُ لَكُمُ ٱلۡإِسۡلَٰمَ دِينٗا

"This day have I perfected your religion for you and fulfill My favor for you, and that Islam be your religion." (QS Al-Maidah:3)

The verse above clearly shows that Islam is the only religion that blessed and true. Islam is a religion that in accordance with human nature and Islam is different from other religions. Islam is a religion that God directly revealed through His apostles prior to the Prophet Muhammad.

Besides uniqueness properties, there are linearity properties. Concerning with linearity properties can be explained that If f and g are Henstock-Kurzweil integrable on $[a,b]$, Then $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

From these explanations are meant essentially to discuss the properties of the translation from one function into several separate functions. So can we associate with religious studies. That is about the translation of Pillars of Faith. We know that Faith consists of six pillars. Faith means believing itself with heart, saying with the tongue, and practice with members of the body, increases with obedience and decreases with disobedience.

The pillars of Faith is the believe in Allah, believe in His angels, believe in His books, faithful to His Apostles, faithful to the Day of Judgement and believe in destiny, good or bad. So clearly shows that Pillars of Faith we can associate with one of properties of Henstock Kurzweil integral is Linearity properties.

Besides pillars of Faith, other example in religious studies about linearity properties of Henstock-Kurzweil integral is Pillars of Islam. Islam is built on five basic Pillars of Islam. Like a house, Pillars of Islam are the pillars or support someone Islamic buildings. In it covered the Islamic laws that govern all aspects of human life. "Verily, Islam is built upon five things: testifying in fact there is no god but Allah and Muhammad is His messenger, establish worship and pay the obligatory alms, fasting and pilgrimage to the House in Ramadan month" (Narrated by Bukhari, Muslim). For anyone who has been working on the five

Pillars of Islam, has not meant that he had a total entry into Islam. He just built a foundation for other charity.

Pillars of Islam is the operational basis of the Five Pillars of Faith. Not enough is said to believe only by doing Pillars of Islam without any attempt to enforce it. Pillars of Islam is training for the Muslims towards Bless Allah. Pillars of Islam consists of Creed, Prayer, Zakat, Fasting and Hajj for those who can afford.

Creed is the agreement between a Muslim with Allah SWT. Someone who has been declared *"La ilaha ilallaah"* means have been ready to fight against all forms God, outside in his life. Prayers are training: an exercise that every Muslim in his life is in order to bow down (worship) to Allah. Zakat is the training, namely as an exercise to give his property, because every property of a Muslim is God's. "You take the zakat from their rich people and you return it to their indigent persons" (HR Mutafaqun 'alahi).

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Fasting is training, that is an exercise in controlling the physical habits, is eating and drinking and spiritual, that is lust. Hajj is the training, namely as an exercise in sacrifice life and property in the way of Allah, to practice unity and equality with other human beings.

So, the explanations above shows that pillars of Faith and pillars of Islam can be the example of properties Henstock-Kurzweil integral in religious studies.

CHAPTER IV
ENCLOSURE

4.1 Conclusion

From the discussion we get conclusion that:

1. δ-*fine* Henstock-Kurzweil partition $D = \{([u_i, v_i], \xi_i)\}_{i=1}^{n}$ we write $S(f, D) = \sum_{i=1}^{n} f(\xi_i)(v_i - u_i)$ where D be a finite collection of interval-point

pairs $\{([u_i, v_i], \xi_i)\}_{i=1}^{n}$, where $\{([u_i, v_i])\}_{i=1}^{n}$ are non-overlapping subintervals of $[a,b]$. Let $\delta(\xi)$ be a positive function on $[a,b]$, $i.e. \delta(\xi):[a,b] \rightarrow \Re^{+}$. And if $\xi_i \in [u_i, v_i] \subset B(\xi_i, \delta(\xi_i)) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1,2,3,...,n$.

2. A function $f:[a,b] \rightarrow \Re$ is said to be Henstock-Kurzweil integrable on $[a,b]$ if there exists a real number S_f such that for every $\varepsilon > 0$ there exists $\delta:[a,b] \rightarrow \Re^{+}$ such that for every δ-*fine* Henstock-Kurzweil partition $D = \{([u_i, v_i], \xi_i)\}_{i=1}^{n}$ of $[a,b]$, we have

$$|S(f,D) - S_f| < \varepsilon.$$

3. And the fundamental properties of Henstock- Kurzweil integral as follows: value of the Henstock- Kurzweil integral is unique, linearity of the Henstock-Kurzweil integral, Additivity of the Henstock-Kurzweil integral, Cauchy criteria, nonnegativity of Henstock-Kurzweil integral, and primitive functio

4.2 Suggestion

Next, the writer realizes that this thesis may be far from being perfect. So, the writer expects that someone will give good comment to make this research become better.

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