A STUDY OF δ-QUASI-BAER RING

THESIS

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Malang, 9th July 2010

Which make statement,

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MOTTO

Tomorrow is another Day, the Day of Struggle

For a better Life

(Soe Hok Gie, 1966)

Destiny is not a matter of Chance, it is a matter of Choice;

It is not a thing to be waited for,

it is a thing to be Achieved

(Willian Jennings Bryan)

DEDICATION

This is my Great and Unforgetable Moment to create this thesis,

this is dedicated to :

My beloved Father and Mother,

Mulyono and Suriyatin

For your endless great loves, cares, sacrifices, advices and prays

May Allah SWT blesses you. Aamiin ...

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Malang, 9th Juli 2010

M. David A. P

ABSTRACT

Putra, Moch. David Andika. 2010. ***A study of δ-Quasi-Baer Ring***. Thesis, Mathematic Department Faculty of Science and Technology The State Islamic University Maulana Malik Ibrahim of Malang.
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Keyword : Derivation, Quasi-Baer ring, characterization, polynomial.

Items that is studied at abstraction algebra basically about set and its operation, and always equivalent with a nonempty set which have elements are operated with one or more binary operation. A set is provided with one or more binary operation is called algebra structure.

Algebra structure with one binary operation which fulfill the certain natures is called group. While for nonempty set with two binary operations which fulfill the certain natures is called ring. At other algebra structure is also studied about Quasi-Baer ring. Let *R* is Quasi-Baer ring if the *right* annihilator of every ideal (as a *right* ideal) is generated by idempotent.

Continuing from Quasi-Baer ring can be developed to become some discussions, those are δ-Quasi-Baer ring. Let δ be a derivation on *R*. A ring *R* is called δ-Quasi-Baer ring if the right annihilator $r_R(X) = \{c \in R \mid dc = 0, \forall d \in X\}$ of every δ-ideal of *R* is generated by idempotent. Let δ : R → R is derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \delta a(b), \forall a,b \in R$, so that the extension of $\delta : R \to R$ is $\bar{\delta}(ab) = \delta(a)b - \delta a(b), \forall a,b \in R$. For a ring *R* with a derivation δ, there exists a derivation on *S* = *R[x; δ]* which extends δ. Considering an inner derivation $\bar{\delta}$ on *S* by *x* defined by $\bar{\delta}(f(x)) = x\,f(x) - f(x)\,x, \ \forall\, f(x) \in S$ .Then,

$$\bar{\delta}(f(x)) = \delta(a_0) + ... + \delta(a_n)x^n \ for\, all \ f(x) = a_0 + ... + a_n x^n \in S$$

$$and\ \bar{\delta}(r) = \delta(r), \forall r \in R$$

which means that $\bar{\delta}$ is an extension of δ. We call such a derivation $\bar{\delta}$ on *S* an extended derivation of δ. For each $a \in R$ and nonnegative integer *n*, there exist

$$t_0,...,t_n \in Z \ such\ that\ x^n a = \sum_{i=0}^{n} t_i \delta^{n-i}(a)x^i .$$

Let *R[x; δ]* is the polynomial ring whose elements are the polynomials denote $\sum_{i=0}^{n} r_i x^i \in R,\ r_i \in R$, where the multiplication operation by $xb = bx + \delta(b),\ \forall\ b \in R$. Three commonly used operations for polynomials are addition “ + “, multiplication “ . ” and composition “ ∘ “. Observe that (*R[x]*, +, .) is a ring and (*R[x]*, +, ∘) is a left nearring where the substitution indicates substitution of *f(x)* into *g(x)*, explicitly *f(x)*∘*g(x)* = *f(g(x)* for each *f(x)*, *g(x)* ∈ *R[x]*.

At other researcher is suggested to perform a research morely about δ-Quasi-Baer ring, with searching natures of others.

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CHAPTER I

INTRODUCTION

1.1 Background of the Study.

In general, some concepts of science discipline have been explained in the Holy Qur'an, one of them is mathematic. The conception from existing mathematics science discipline in the Holy Qur'an among others is the problem of logic, modeling, statistic, set, group, ring, and others.

Algebra is a branch of mathematic which is very complex science of items in it. One of them is abstraction algebra, which studying with kinds of algebraic structure was accompanied by some axioms that defined to this.

Algebraic structure always involves 3 elements that are a set is not empty, one or more binary operations, and some axioms. The number of axioms and operations are going into effect to become distinguishing between abstract algebra which is one with another algebra structure (Arifin, 2000). For example, the Group is an algebraic structure which was built by a single operation, fulfilling the nature of closed, associative, owning identity element, and each element has the inverse. While the study of the set with one binary operation (group) in Islam is the concept that human being is created in pairs. Paying attention of word Allah SWT in letter of Al-Faathir sentence 11:

وَٱللَّهُ خَلَقَكُم مِّن تُرَابٍ ثُمَّ مِن نُّطۡفَةٍ ثُمَّ جَعَلَكُمۡ أَزۡوَٰجًاۚ وَمَا تَحۡمِلُ مِنۡ أُنثَىٰ
وَلَا تَضَعُ إِلَّا بِعِلۡمِهِۦۚ وَمَا يُعَمَّرُ مِن مُّعَمَّرٖ وَلَا يُنقَصُ مِنۡ عُمُرِهِۦٓ إِلَّا فِي
كِتَٰبٍۚ إِنَّ ذَٰلِكَ عَلَى ٱللَّهِ يَسِيرٞ ﴿١١﴾

Its meaning:

"And Allah create you of land later from sperma, then He (Allah) make you of pairing (woman and men) and there's no a pregnant woman and do not bear but with an Acknowledgement Allah SWT and is not on any account lengthened by age a long lived and do not is also lessened by its age, but (have been specified) in Lauhil Mahfudzh. Real like that to Allah it is so easy".

From word Allah above that human being is pairing that are men and woman.

Algebraic structures with two binary operations which fulfill certain conditions are called Ring. While study set with two binary operations (ring) in Islam concept that is, that human being is created by pairs and way of pairing it with certain laws. Paying attention of word Allah SWT in letter of An-Nisaa' sentence 23:

حُرِّمَتۡ عَلَيۡكُمۡ أُمَّهَٰتُكُمۡ وَبَنَاتُكُمۡ وَأَخَوَٰتُكُمۡ وَعَمَّٰتُكُمۡ وَخَٰلَٰتُكُمۡ وَبَنَاتُ
ٱلۡأَخِ وَبَنَاتُ ٱلۡأُخۡتِ وَأُمَّهَٰتُكُمُ ٱلَّٰتِيٓ أَرۡضَعۡنَكُمۡ وَأَخَوَٰتُكُم مِّنَ
ٱلرَّضَٰعَةِ وَأُمَّهَٰتُ نِسَآئِكُمۡ وَرَبَٰٓئِبُكُمُ ٱلَّٰتِي فِي حُجُورِكُم مِّن نِّسَآئِكُمُ
ٱلَّٰتِي دَخَلۡتُم بِهِنَّ فَإِن لَّمۡ تَكُونُواْ دَخَلۡتُم بِهِنَّ فَلَا جُنَاحَ عَلَيۡكُمۡ

وَحَلَٰٓئِلُ أَبۡنَآئِكُمُ ٱلَّذِينَ مِنۡ أَصۡلَٰبِكُمۡ وَأَن تَجۡمَعُواْ بَيۡنَ ٱلۡأُخۡتَيۡنِ إِلَّا
مَا قَدۡ سَلَفَۗ إِنَّ ٱللَّهَ كَانَ غَفُورٗا رَّحِيمٗا ﴿٢٣﴾

Its meaning:

"Illicit of you (you was married) your children which is woman; your brothers and sisters which is woman, your father brothers and sisters which is woman; your mother brothers and sisters which is woman; woman children of your brothers and sisters which is men; woman children of your brothers and sisters which is woman; your which suckle you; sister of brother; your wife mothers (parent in law); your wife children which in your conservancy from your wife which have meddle, but if you not yet mixed with your wife is (and you have divorce), hence yours innocent marrying it (and illicit for you) child wife contain you (daugter; son in law); and accumulate (in marriage) two woman which you is, except which have happened at is old world; In Fact Forgiving Allah The most again The most Humane".

Hence from word Allah above that human being is pairing between woman and men with marriage. However, procedures get married with its couple have to judicially religion (Islam).

At certain range of time natural ring of growth to the last peep out new algebra structure, called Baer ring. Conception in Baer ring very different from concept of ring, because axiom going into effect is unequal, more at development of new axioms, so that peep out various interesting problems to be studied furthermore along growth of science, specially abstraction algebra discipline.

At theory of Baer ring explained as ring which each right annihilator of Principal Ideal which generated by an idempotent. For example, if R is a commutative ring Von Nouman, then R is p. q-Baer, but is not Quasi-Baer. A ring fill a common form of Rickart' condition (every Annihilator right of an element is generated (as a right ideal) by the idempotent) has a characteristic as a Baer ring. Related to statement above, studying Baer ring is one of the items at abstraction algebra expanding and getting attention.

By reviewing and analyzing the Baer ring, we can extract a formulation that will further simplify the process of applying to the real world.

Like explained that at Baer Ring studied by about Quasi-Baer ring and one of the interesting topics it is study of Quasi-Baer ring. Therefore, hence here writer interest to study about the characteristic of δ-Quasi-Baer ring and Derivation Polynomials over Quasi-Baer ring untitled "***A Study of δ-Quasi Baer Ring***".

1.2 Statement of the Problem

Based on the description of the background of the study above, the researcher formulated the following problems:

1. What is δ-Quasi-Baer ring?
2. How is Characterization of δ-Quasi-Baer ring?
3. How is Derivation Polynomials over Quasi-Baer ring?

1.3 Scope and Limitation

In this thesis, the problems of this discussion will be limited into definitions and theorems of δ-Quasi-Baer ring, Characterization of δ-Quasi-Baer ring, and Derivation Polynomials over Quasi-Baer ring.

1.4 Objectives of the Study

Concerning the problems of the study above, the objectives of the study are:

1. To find the conclusion of meaning of δ-Quasi-Baer ring.
2. To describe the Characterization of δ-Quasi-Baer ring.
3. To get the description of Derivation Polynomial over Quasi-Baer ring.

1.5 Significance of the Study

The result of research is expected to discuss this issue benefits for:

1. For Author
   a. Adding knowledge and science on matters relating to the δ-Quasi-Baer ring.
   b. Developing scientific insights about the description of the δ-Quasi-Baer ring.
2. For Institutions
   a. Giving information about instructional materials abstract algebra.
   b. Giving addition to the literature.
3. For Students: As information material for further study of abstract algebra at the δ-Quasi-Baer ring.

1.6 Research Methods

The method used in this research is the literature method (library research) or a literature review, which conducts research to obtain data and some information as well as objects used in the discussion of the problem.

Literature study is the appearance of argument to explain the results of scientific reasoning process to think about an issue or topic discussed in the literature review of this research. As the steps that will be used by researcher in studying this research are as follows:

1. Finding the main literature that was used as reference in this discussion. Literature is "David S. Dummit and Richard M. Foote. 1991. *Abstract Algebra*. Prentice-Hall, Inc., Raisinghania. 1980. *Modern Algebra*. S. Chand and Company, Ltd., S. K. Berberian. 1988. *Baer and Baer *- Ring*. The University of Texas at Austin".
2. Collecting a variety of supporting literature, both sourced from books, journals, articles, lectures and training, internet, and other related issues will be discussed in this research.
3. Describing the concept of Quasi-Baer ring.
4. Applying the concept of Quasi-Baer ring to describe the characteristics of δ-Quasi-Baer Ring and derivation polynomials over Quasi Baer Ring with the following steps:
   a. Determining the definition associated with the Quasi-Baer ring, then gives example from that definition.
   b. Determining the theorem relating to the Quasi-Baer ring, then proving the theorem.
   c. Making conclusions (new principles) of δ-Quasi-Baer ring, from a combination of definitions and theorems that have been there.

d. Explaining the characteristics of δ-Quasi-Baer ring then gives example.

e. Explaining derivation polynomials over Quasi-Baer ring then gives example.

1.7 Writting Systematic

In order for the reader to read the results of this study were easy to understand and do not find trouble, then in written according to a systematic presentation of the outline is divided into four chapters, namely:

CHAPTER I INTRODUCTION

Introduction includes: Background of the Study, Statement of the Problem, Scope and Limitation, Objectives of the Study, Significance of the Study, Research Methods and Writing Systematic.

CHAPTER II REVIEW OF RELATED LITERATURE

In this section consists of several concepts (theories) that supports the discussion of among others, discusses the Ring, Sub-Ring, Ideal, the Polynomial Ring, Characteristic of Ring, Near Ring, Baer Ring and Studies in Religion.

CHAPTER III DISCUSSION

This discussion contains δ-Quasi-Baer ring, the characteristic of δ-Quasi-Baer ring, and derivation polynomials over Quasi-Baer ring.

CHAPTER IV ENCLOSURE

In this chapter will discuss the conclusions and recommendations.

CHAPTER II

REVIEW OF RELATED LITERATURE

2.1 Ring

An algebraic structure which there of a non-empty set with one binary operation is called group. This system of algebra is not enough to accomodate structures in mathematic. In this section is developed a system of algebra which there of a non-empty set with two binary operations is called ring R.

Definition

1. A ring R is a set together with two binary operations addition (first operation) and multiplication (second operation) satisfying the following axioms:
   a. $(R, +)$ is abelian group
   b. Multiplication is associative: $(a \times b) \times c = a \times (b \times c)$;
      for all $a, b, c \in R$,
   c. The distributive laws hold in R : for all $a, b, c \in R$,
      $(a+b) \times c = a \times c + b \times c$ and $a \times (b+c) = a \times b + a \times c$.
2. The ring R is commutative if multiplication is commutative.
3. The ring R is said to have an identity (or contain *a.I*) if there is an element $I \in R$
   with $I \times a = a \times I = a$; for all $a \in R$
   (Dummit and Foote, 1991: 225).

We shall usually write simply $a.b$ *rather than* $a \times b$ *for* $a,\ b \in R$. The additive identity of R we always be denote by *0* and the additive inverse of the ring element a will be denote by $(-a)$.

The condition that R be a group under addition is a fairly natural one, but it may seem artificial to require that this group be abelian. One motivation for this is that if the ring R has *a.1*, the commutativity under addition is forced by the distributive laws. To see this, compute the product $(1+1)(a \times b)$ in two different ways, using the distributive laws (but not assuming that addition is commutative).

One obtains

$$(1+1)(a+b) \ = \ 1(a+b)+1(a+b) \ = \ 1a+1b+1a+1b \ = \ a+b+a+b$$

and

$$(1+1)(a+b) \ = \ (1+1)a+(1+1)b \ = \ 1a+1a+1b+1b \ = \ a+a+b+b\,.$$

Since R is group under addition, this implies $b+a=a+b$, that R under addition is necessarily commutative. Fields are one of the most important examples of ring.

Example. Show that if $(Z, +, \times)$ with Z as integer number is a ring R!

Answered:

1. $(Z, +)$ is abelian group such that

   i. If $a, b \in Z$, so that $a+b \in Z$. Hence Z is closed by additive operation.

ii. If $a, b, c \in Z$, so that $(a+b)+c = a+(b+c)$. Hence additive operation is associative in *Z*.

iii. $\exists 0 \in Z$, *such that* $a+0=0+a=a$, $\forall a \in Z$. Hence 0 is an identity of additive.

iv. $\forall a \in Z$, $\exists\, (-a) \in Z$, *such that* $a+(-a)=(-a)+a=0$.

Hence inverse of *a* is (-*a*).

v. Additive operation is commutative in *Z*

$\forall\, a, b \in Z$, *such that* $a+b=b+a$.

2. Multiplication is associative in *Z*

$(a \times b) \times c = a \times (b \times c)$, $\forall a, b, c \in Z$.

3. Multiplication is distributive of additive

$(a+b) \times c = (a \times c) + (b \times c)$, $\forall a, b, c \in Z$

$a \times (b+c) = (a \times b) + (a \times c)$, $\forall a, b, c \in Z$.

2.1.1 Commutative Ring

Definition. Let (*R*, +, $\times$) is commutative ring of *R* if :

I. (*R*, +) is commutative group

II. (*R*, $\times$) is commutative semigroup

III. Multiplication ($\times$) is disributive of additive (+)

(Hidayanto and Irawati, 2000: 8)

The simplest examples of rings are the trivial rings obtained by taking *R* to be any commutative group (denoting the group operation by +) and the defining the multiplication $\times$ $on\ R\ by: a \times b = 0\ for\ all\ a,b \in R.$ It is easy to see that this multiplication defines a commutative ring. In another particular, if *R = {0}* is the trivial group, the resulting ring *R* is called the zero ring, denote *R = 0*. Except for the zero ring, a trivial ring does not contain an identity (*R = 0* is the only ring where 1 = 0; we shall often exclude this ring by imposing the condition $1 \neq 0$).

Although trivial rings have two binary operations, multiplication adds no new structure to the additive group and the theory of rings gives no information which could not already be obtained from (abelian) group theory.

Example. Show that (*R*, +, $\times$) is a ring *R*, if for each $x \in R$ there is $x^2 = x$, so that *R* is a commutative ring!

Answered

Choose for each $a, b \in R$

$$(a+a)^2 = a^2 + aa + a^2$$

$$(a+a) = a+a+a+a$$

$$(a+a) = (a+a)+(a+a)$$

$$(a+a)-(a+a) = (a+a)$$

$$a+a=0$$

$$a=(-a)$$

And so that,

$$(a+b)^2 = a^2 + ab + ba + b^2$$

$$(a+b) \; = a + ab + ba + b$$

$$(a+b) \; = (a+b) + ab + ba$$

$$(a+b) - (a+b) = ab + ba$$

$$0 = ab + ba$$

$$-ab = ba$$

$$(-a)b = ba \qquad (\therefore with\ a = -a)$$

$$ab = ba$$

2.1.2 Ring with Unity

Definition. Let $(R,\ +,\ \otimes)$ be a ring R. If there is $x \in R,\ such\ that\ x \otimes y = y \otimes x = y.$ So x is called unity element in R and writed 1. So that ring which there is unity element is called ring with unity (Hidayanto and Irawati, 2000: 11).

Example. Show that $(R,\ +,\ \times)$ with real number R is ring with unity!

Answered

A. $(R, +)$ is abelian group, such that:

1. Choose $a, b \in R$, so that $a + b \in R$. Hence R is closed at additive operation.

2. Choose $a, b, c \in R$, so that $(a+b)+c = a+(b+c)$. Hence additive operation is associative in R.

3. $\exists\, 0 \in R$ such that $a+0 = 0+a = a, \forall a \in R$. Hence 0 is additive identity.

4. $\forall a \in R, \exists (-a) \in R, \ni a+(-a) = (-a)+a = 0$. So inverse of a is $-a$.

5. Additive operation is commutative in R

   $\forall a, b \in R$, *so that* $a+b = b+a$.

B. Multiplication operation is associative in R

   $(a \times b) \times c = a \times (b \times c), \forall a, b, c \in R$.

C. Multiplication operation is distributive of additive

   $(a+b) \times c = (a \times c)+(b \times c), \forall a, b, c \in R$

   $a \times (b+c) = (a \times b)+(a \times c), \forall a, b, c \in R$

D. There is unity element

   Let $a \in R$, so that $a \times b = b \times a = b$.

   Such that, its unity element is 1, it mean that $a = 1$. Clearly, $(R, +, \times)$ is ring with unity.

The Properties of Ring

2.1.3 Definition. Let $(R, +, \times)$ be a ring R and let $a \in R$ with $a \neq 0$ a is called zero divisor if there is $b \neq 0$ so that $a \times b = 0$ *or* $b \times a = 0$ (Pinter, 1990: 173).

Example. Let $(Z_8, +, \times)$ be a ring *R* of arbitrary integer set. Show that zero divisor from Z_8.

Answered. $Z_8 = \{\overline{0}, \overline{1}, \overline{2}, ..., \overline{7}\}$

$\overline{0} = \overline{8} = \overline{16} = \overline{24} = \overline{32} = \overline{40} = \overline{48} = \overline{56} = \overline{64}$

$\overline{2}$ is zero divisor because $\exists\, \overline{2} \ni \overline{2} \times \overline{4} = \overline{8} = 0$

$\overline{4}$ is zero divisor because $\exists\, \overline{4} \ni \overline{4} \times \overline{2} = \overline{8} = 0$

Hence, zero divisor of Z_8 is $\overline{2}$ and $\overline{4}$.

2.1.4 Definition. Let *R* be a ring.

Assume *R* has an identity $1 \neq 0$. An element *u* of *R* is called a unit in *R* if there is some *v* in *R* such that *uv = vu = 1*. The set of units in *R* is denoted R^x. It is easy to see that the units in a ring *R* form a group under multiplication so R^x will be referred to as the group of units of *R*. In this terminology a field is a commutative ring *F* with identity $1 \neq 0$ in which every nonzero element is a unit, $F^x = F - \{0\}$. Observe that a zero divisor can never be a unit. Suppose for example that a is a unit in *R* and that $ab = 0$ for some non-zero b in *R*. Then $va = 1$ *for some* $v \in R$, so $b = 1 \times b = (v \times a) \times b = v \times (a \times b) = v \times 0 = 0$, a contradiction. Similarly, if $ba = 0$ for some nonzero b then a cannot be a unit. This showing is in particular that fields contain no zero divisors (Dummit and Foote, 1991: 227).

2.1.5 Examples:

1. If R is the ring of all functions from the closed interval [0, 1] to R then the units of R are the functions that are not zero at any point (for such f its inverse is the function $\frac{1}{f}$). If f is not a unit and not zero then f is a zero divisor because if we define

$$g(x) = \begin{cases} 0, & if \quad f(x) \neq 0 \\ 1, & if \quad f(x) = 0 \end{cases}$$

then g is not the zero function but $f(x)g(x)=0$ *for all x.*

2. If R is the ring of all continuous functions from the closed interval [0, 1] to R then the unit of R are still the functions which are not zero at any point, but now there are functions that are neither units nor zero divisor. For instance, $f(x)=x-\frac{1}{2}$ has only one zero (at $x=\frac{1}{2}$) so f is not unit. On the other hand, if $gf=0$ then g must be zero for all $x\neq\frac{1}{2}$, and the only continuous function with this property is the zero function. Hence f is neither a unit nor a zero divisor. Similarly, no function with only a finite (or countable) number of zeros on [0, 1] is a zero divisor. This ring also contains many zero divisors. For instance let

$$f(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{2}, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

and let $g(x) = f(1-x)$. Then f and g are nonzero continuous functions whose product is the zero function.

2.1.6 Definition. A commutative ring with identity $1 \neq 0$ is called an integral domain if it has no zero divisors.

The absence of zero divisors in integral domains gives these rings a cancellation property (Dummit and Foote, 1991: 228).

2.1.7 Proposition. Assume that a, b *and* c are elements of any ring with a not a zero divisor. If $ab = ac$ then either $a = 0$ *or* $b = c$ (if $a \neq 0$ we can cancel the a's). In particular, if a, b, c are any elements in an integral domain and $ab = ac$, then either $a = 0$ *or* $b = c$.

Proof. If $ab = ac$ then $a(b-c) = 0$ so either $a = 0$ *or* $b - c = 0$. The second statement follows from the first and the definition of an integral domain.

2.1.8 Idempotent

The idempotent is same with meaning of projection. An idempotent p is called a projection if $p = p^*$ (where * is an involution).

Example. Let $(M_5, +, \times)$ be a ring R of arbitrary integer set in modulo 5. If we consider the set $M_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$, then M_5 is a ring with respect to addition modulo 5 and multiplication modulo 5. Here *0* is the identity for $+_5$.

Answered. $M_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$

$\bar{0}=\bar{5}=\overline{10}=\overline{15}=\overline{20}=\overline{25}=\overline{30}=\overline{35}=\overline{40}$

Let p is idempotent, such that

$\rightarrow p^2=0$

$p\times p=0$

$\bar{0}$ is idempotent because $\exists\,\bar{0} \ni \bar{0}\times\bar{0}=\bar{0}$

$\bar{1}$ is not idempotent because $\exists\,\bar{1} \ni \bar{1}\times\bar{1}=\bar{1}\neq\bar{0}$

$\bar{2}$ is not idempotent because $\exists\,\bar{2} \ni \bar{2}\times\bar{2}=\bar{4}\neq\bar{0}$

$\bar{3}$ is not idempotent because $\exists\,\bar{3} \ni \bar{3}\times\bar{3}=\bar{9}=\bar{4}\neq\bar{0}$

$\bar{4}$ is not idempotent because $\exists\,\bar{4} \ni \bar{4}\times\bar{4}=\overline{16}=\bar{1}\neq\bar{0}$

Hence, idempotent of M_5 is $\bar{0}$.

2.2 Sub-Ring

Let S be a non-empty subset of a ring R, we say that S is stable for the addition and multiplication compositions in R, if $a\in S,\, b\in S \Rightarrow a+b\in S$ and $a\in S,\, b\in S \Rightarrow a\times b\in S$.

Also these compositions restricted on points of S only are said to be the induced compositions.

Definition. A non-empty subset S of a ring R is called a *sub-ring* of the ring R, if S is stable for the compositions in R and S it self is a ring with respect to the induced compositions (Raisinghania, 1980: 356).

Since every set is a subset of it self. So if R is a ring, then R is surely a sub-ring of R. Also, if 0 be the zero of a ring R, then the subset $S = \{0\}$ is clearly a sub-ring of R.

Thus every ring R has surely two sub-rings namely R and $\{0\}$. These sub-rings are called trivial or improper sub-rings. A sub-ring other than these sub-rings is called a proper sub-ring.

Example 1. The set of all rational numbers is a sub-ring of the ring of all real numbers and set of all real numbers is a sub-ring of the ring of all complex numbers.

Example 2. The set of all $n \times n$ *diagonal matrices* with their elements as real numbers is a sub-ring of the ring of all $n \times n$ *matrices* their elements as real numbers.

Definition. Let $(R, +, \times)$ be a ring R, if $S \subset R, S \neq \emptyset$ and $(S, +, \times)$ is a ring, so that S is called sub-ring of R (Hidayanto and Irawati, 2000: 30).

Example. Let $(R, +, \times)$ be a ring R, $S = \{a \in R \mid x \times r = r \times x, \forall r \in R\}$. Show that S is sub-ring of R.

Answered

I. $S \neq \emptyset$, *such that* $0 \times r = r \times 0 = 0, \forall r \in R$, *then* $0 \in S$.

II. $x \in S$, *such that* $x \in R$, *then* $S \subseteq R$

III. Choose $x, y \in S$, so that $x \times r = r \times x$ *and* $y \times r = r \times y$

Will be proved $(x - y) \times r = r \times (x - y), \forall r \in R$

Choose any $r \in R$, such that

$$(x-y)\times r=(x\times r)-(y\times r)$$

$$=(r\times x)-(r\times y)$$

$$=r\times(x-y)$$

$Hence,\ x-y\in S$

IV. Choose $x, y \in S$, so that $x\times r=r\times x\ and\ y\times r=r\times y$

$$(x\times y)\times r=x\times(y\times r)$$

$$=x\times(r\times y)$$

$$=(x\times r)\times y$$

$$=(r\times x)\times y$$

$$=r\times(x\times y)$$

$Hence,\ x\times y\in S\ and\ S\ is\ sub-ring\ of\ R.$

Conditions for a sub-ring

Theorem 1. The necessary and sufficient conditions for non-empty subset S of a ring R to be a sub-ring there of are that $a\in S,\ b\in S\Rightarrow a-b\in S,$ and $a\in S,\ b\in S\Rightarrow a\times b\in S.$

Proof. **The Conditions are necessary**. Let *S* be a sub-ring of a ring *R*.

Now, if *a* and *b* are any two elements of *S*, then

$a\in S,\ b\in S\Rightarrow a\in S,\ (-b)\in S,$ [so *S* being a ring, it is a group for addition]

$\Rightarrow a+(-b) \in S$, [so S is stable for addition]

$\Rightarrow a-b \in S$, [so $a + (-b) = a - b$]

Also $a \in S,\ b \in S \Rightarrow a \times b \in S,$ [so S is stable for multiplication]

The Conditions are sufficient. Let S be a non-empty subset of a ring R such that,

$a \in S,\ b \in S \Rightarrow a-b \in S,$ and $a \in S,\ b \in S \Rightarrow a \times b \in S.$

Using these properties we have,

$$a \in S,\ a \in S \Rightarrow a-a \in S$$

$$\Rightarrow 0 \in S$$

and

$$0 \in S,\ a \in S \Rightarrow 0-a \in S$$

$$\Rightarrow (-a) \in S$$

Consequently,

$$a \in S,\ b \in S \Rightarrow a \in S, (-b) \in S$$

$$\Rightarrow a-(-b) \in S$$

$$\Rightarrow a+b \in S.$$

Also as given,

$$a \in S,\ b \in S \Rightarrow a \times b \in S.$$

Thus S is stable for addition as well as multiplication compositions, the additive identity exists in S and every element in S has its negative in S.

Also S being a subset R, each element of S is also an element of R and since addition is commutative and associative, multiplication is associative and distributes addition in R, all these properties are therefore also satisfied by elements of S.

Hence S is a sub-ring of R.

Theorem 2. The necessary and sufficient conditions for a non-empty subset S of a ring R to be a sub-ring are,

(i) $S+(-S)=S$

(ii) $S\times S\subseteq S$

where $(-S)$ is the set consisting of negatives of all elements of S.

Proof. **The Conditions are necessary**. Let S be a sub-ring of a ring R, then clearly S is a subgroup of the additive group R.

Now, let $a+(-b)$ be an arbitrary element of $S+(-S)$, then

$$a+(-b)\in S+(-S)\Rightarrow a\in S,(-b)\in(-S)$$

$$\Rightarrow a\in S,\, b\in S$$

$$\Rightarrow a-b\in S \qquad [\because S \text{ is a sub}-\text{ring}]$$

$$\Rightarrow a+(-b)\in S$$

So that, $S+(-S)\subseteq S$

Again, if a be an arbitrary element of S, then we may write,

$$a = a + 0 = a + (-0) \in S + (-S).$$

So that, $a \in S \Rightarrow a \in S + (-S)$

and therefore, $S \subseteq S + (-S)$.

Hence, $S + (-S) = S$.

Also S being a sub-ring, it is stable for multiplication.

Therefore,

$a \times b \in S \times S \Rightarrow a \in S,\ b \in S$

$$\Rightarrow a \times b \in S.$$

Hence, $S \times S \subseteq S$.

The Conditions are sufficient. Let S be a non-empty subset of a ring R such that, $S + (-S) = S$ *and* $S \times S \subseteq S$.

Then,

$a \in S,\ b \in S \Rightarrow a \in S, (-b) \in (-S)$

$$\Rightarrow a + (-b) \in S + (-S)$$

$$\Rightarrow a - b \in S + (-S)$$

$$\Rightarrow a - b \in S \qquad [\because S + (-S) = S]$$

Also, $a \in S,\ b \in S \Rightarrow a \times b \in S \times S$

$$\Rightarrow a \times b \in S \quad [\because S \times S) \subseteq S]$$

Thus, $a \in S,\ b \in S \Rightarrow a - b \in S$ *and* $a \times b \in S$

which are the necessary and sufficient conditions for S to be a sub-ring.

Hence S is a sub-ring of ring R.

2.3 Characteristic of a Ring

The characteristic of a ring R is the smallest positive integer n, if exists, such that,

$$na = 0 \; \forall \, a \in R$$

where 0 denotes the zero of R, the additive identity of the group $(R, +)$.

In case such an n does not exist, we say that the ring R is of characteristic zero or of infinite characteristic (Raisinghania, 1980: 330).

Example: If we consider the set $I_7 = (0, 1, 2, 3, 4, 5, 6)$, then I_7 is a ring with respect to addition modulo 6 and multiplication modulo 6. Here 0 is the identity for $+_6$.

Answered

It is clear in this case that 7 is the least positive integer for which

$$7 \times_6 a = 0 \; \forall \, a \in I_7$$

And there is no positive integer $r < 7$ for which

$$r \times_6 a = 0 \; \forall \, a \in I_7$$

Hence the ring $(I_7, +_6, \times_6)$ is of characteristic 6.

As an another example, if we consider the ring I of all integers then we observe that there is no positive integer n for which

$$na = 0 \; \forall \, a \in I$$

So that I is of infinite or zero characteristic.

2.4 Ideals

Definition

A sub-ring S of a ring R is called a

(i) right ideal of R if

$a \in S,\ \ r \in R\ \ \Rightarrow\ \ a \times r \in S$

(ii) left ideal of R if

$a \in S,\ \ r \in R\ \ \Rightarrow\ \ r \times a \in S$

(iii) both sided ideal or simply an ideal if S is a right ideal as well as a left ideal if $a \in S,\ \ r \in R\ \ \Rightarrow\ \ a \times r \in S\ \ \ and\ \ \ r \times a \in S$

(Raisinghania, 1980: 361).

Obviously, $\{0\}$ and R are ideals of any ring R. These are referred to as "*trivial*" or "*improper*" ideals. All ideals of R, other than $\{0\}$ and R are called *proper ideals*.

A ring having no proper ideals is called simple ring. It is clear from the above definitions that the left ideal of a commutative ring is also the right ideal of the ring.

2.4.1 Theorem

The necessary and sufficient conditions for a non-empty subset S of a ring R to be an ideal of R are

i) $a \in S,\ \ b \in S\ \ \Rightarrow\ \ a - b \in S$

ii) $a \in S,\ r \in R \ \Rightarrow\ a \times r \in S \ \ and \ \ r \times a \in S$

Proof. The Conditions are necessary. Let S be an ideal of a ring R, so that by definition of an ideal, S is a sub-ring of R such that

$a \in S,\ r \in R \ \Rightarrow\ a \times r \in S \ \ and \ \ r \times a \in S$.

In particular, S is a sub-group of the additive group R such that

$a \in S,\ r \in R \ \Rightarrow\ a \times r \in S \ \ and \ \ r \times a \in S$.

But the necessary and sufficient condition for a non-empty subset S to be a sub-group of the additive group R being

$a \in S,\ b \in S \ \Rightarrow\ a - b \in S$.

Consequently,

$a \in S,\ b \in S \ \Rightarrow\ a - b \in S$

and

$a \in S,\ r \in R \ \Rightarrow\ a \times r \in S \ \ and \ \ r \times a \in S$.

Hence the conditions are necessary.

The Conditions are sufficient. Let S be a non-empty subset of a ring R given in such a way that

i) $a \in S,\ b \in S \ \Rightarrow\ a - b \in S$

ii) $a \in S,\ r \in R \ \Rightarrow\ a \times r \in S \ \ and \ \ r \times a \in S$.

Now, if a and b be any two arbitrary elements of S, then by condition (i),

$a \in S,\ b \in S \ \Rightarrow\ a - b \in S$.

Also by condition (ii),

$a \in S,\ b \in S \ \Rightarrow\ a \in S,\ b \in R \ \Rightarrow\ a \times b \in S$

Thus,

$a \in S,\ b \in S \ \Rightarrow\ a - b \in S \quad and \quad a \times b \in S.$

Which are necessary and sufficient conditions for S to be a sub-ring.

Consequently, S is a sub-ring of R such that

$a \in S,\ r \in R \ \Rightarrow\ a \times r \in S \quad and \quad r \times a \in S$.

Hence S is an ideal of R, showing there by that the conditions are sufficient.

2.4.2 Principal Ideal

An ideal of a ring generated by a single element of a ring is called a *Principal Ideal* (Raisinghania, 1980: 363).

2.4.3 Principal Ideal Ring

A commutative ring R without zero divisors and with unit element is called a *Principal Ideal Ring*, if every ideal of R is a Principal Ideal (Raisinghania, 1980: 363).

2.4.4 Maximal Ideal

An ideal S of a Ring R such that $S \neq R$ is called a *Maximal Ideal* of R if there exists no proper ideal of R containing S (Raisinghania, 1980: 363).

2.4.5 Example 1:

(i) Let Q be the ring of all rational numbers and let I be the set of all integers. Then clearly I is sub-ring of Q. But I is neither a left ideal of Q nor a right ideal of Q, since neither the product of an integer and a rational number is essentially an integer nor the product of a rational number and an integer is essentially an integer.

(ii) The set Q of all rational numbers is a sub-ring of the ring R of real numbers, but it is neither a left ideal nor a right ideal of R, since $a \in Q,\ r \in R$ not necessarily implies that $r \times a \in Q$ and $a \in Q,\ r \in R$ not necessarily implies that $a \times r \in Q$.

2.4.6 Example 2:

Let R be the ring of all matrices of the type 2×2 with their elements as integers and let S be the set of all 2×2 matrices of the form $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ with their elements as integers.

Then for any two matrices

$$A = \begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} a_2 & 0 \\ b_2 & 0 \end{pmatrix} \text{ in } S,$$

$$A - B = \begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix} - \begin{pmatrix} a_2 & 0 \\ b_2 & 0 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 & 0 \\ b_1 - b_2 & 0 \end{pmatrix} \in S$$

$[\because\ a_1, a_2\ \ and\ \ b_1, b_2\ \ being\ \ integers,\ so\ \ are\ \ a_1 - a_2\ \ and\ \ b_1 - b_2]$

and

$$AB = \begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ b_2 & 0 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & 0 \\ b_1 b_2 & 0 \end{pmatrix} \in S$$

$[\because\ a_1 a_2\ \ and\ \ b_1 b_2\ \ are\ \ integers]$

Thus S is a sub-ring of R.

Now, if $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an arbitrary member of R

and

$A = \begin{pmatrix} p & 0 \\ q & 0 \end{pmatrix}$ is any arbitrary member of S, then

$$T \times A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} p & 0 \\ q & 0 \end{pmatrix} = \begin{pmatrix} ap+bq & 0 \\ cp+dq & 0 \end{pmatrix} \in S$$

$[\because \; ap+bq \;\; and \;\; cp+dq \;\; are \;\; integers \;\; whenever \;\; so \;\; are \;\; a, \; b, \; p, \; q]$

Thus

$$A \in S, \;\; T \in R \Rightarrow T \times A \in S$$

So S is a left ideal for R.

Again, if we consider a member

$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ of S and a member,

$T = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ of R, then

$$A \times T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \notin S$$

Thus

$$A \in S, \;\; T \in R \;\; but \;\; A \times T \notin S.$$

Hence S is not a right ideal for R.

Accordingly, S is a left ideal but not a right ideal for R.

2.5 Polynomial Ring

Definition

Let R be a Ring. Then an infinite ordered set $(a_0, a_1, \ldots, a_n, \ldots)$ of elements of R with at most a finite number of non-zero elements is called a polynomial over a ring R (Raisinghania, 1980: 422).

In other words an infinite sequence $(a_0, a_1, \ldots, a_n, \ldots)$ of elements of R is said to be a polynomial over R if there exists a non-negative integer n such that $a_i = 0$, the zero of the ring $R \quad \forall \quad i > n$. We denote such a polynomial by the symbol

$$f(x) = a_0 x^0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + 0x^{n+1} + 0x^{n+2} \ldots = \sum_{i=0}^{\infty} a_i x^i$$

Where the symbol x is not an element of R and is known as an indeterminate over the ring R. We call $a_0 x^0, a_1 x^1, a_2 x^2, \ldots, a_n x^n, \ldots$ etc, as the terms of this polynomial *f(x)* and the elements $(a_0, a_1, \ldots, a_n, \ldots)$ of R as the coefficients of these terms. The different powers of x simply indicate the ordered place of the different coefficients and the symbol '+' in between the different terms indicates the separation of the terms.

2.5.1 Degree of a Polynomial.

If $f(x) = a_0 x^0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$ be a polynomial over the ring R such that $a_n \neq 0$ and $a_i = 0 \quad \forall \quad i > n,$ then such a polynomial is said to be a polynomial of degree n and we write deg. *f(x)* = n. Also in this case $a_n x^n$ is called the leading term of the polynomial and a_n is called the leading coefficient. We agree to write such a polynomial of degree *n* as

$$f(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 x^0 + a_1 x + a_2 x^2 + \ldots + a_n x^n.$$

If $a_i = 0 \quad \forall \quad i > n$. Then f(x) is called a zero polynomial. We never define the degree of a zero polynomial.

The polynomial *f(x)* consisting of a single term $a_0 x^0$ is called a constant polynomial and such a polynomial is said to have the degree zero. Also a polynomial *f(x)* over a ring *R* with unity is called a monic polynomial if the coefficient of the leading term is the unity of *R*.

2.5.2 Equality of Polynomials Two polynomials

$$f(x) = a_0 x^0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \ldots$$

and

$$g(x) = b_0 x^0 + b_1 x + b_2 x^2 + \ldots + b_n x^n \ldots$$

over a ring *R* are said to be equal if $a_i = b_i$ for every integer $i \geq 0$.

Set of all polynomials over a ring *R* in an indeterminate *x*. Let *R* be a ring and let us denote by *R[x]* the set of all polynomials over the ring *R* in a determinate *x*, then we define the addition and multiplication of polynomials in *R[x]* as under:

2.5.3 Addition of Polynomials

Let $f(x) = a_0x^0 + a_1x + a_2x^2 + a_3x^3 ... + a_nx^n + ...$

and $g(x) = b_0x^0 + b_1x + b_2x^2 + b_3x^3 ... + b_nx^n + ...$

be any two polynomials in *R[x]*, then the sum of *f(x)* and *g(x)* denote by *f(x)+g(x)* is defined by

$f(x) + g(x) = (a_0 + b_0)x^0 + (a_1 + b_1)x + (a_2 + b_2)x^2 + ... + (a_n + b_n)x^n + ...$

which is clearly a polynomial over the ring and therefore a member of *R[x]*.

We may note here that if *f(x)* and *g(x)* be any two non-zero polynomials over a ring *R*, then

deg. [*f(x)* + *g(x)*] $\leq$ max. [*deg. f(x), deg. g(x)*],

for, if

$f(x) = a_0x^0 + a_1x + a_2x^2 + a_3x^3 ... + a_nx^n,\ a_n \neq 0$

and $g(x) = b_0x^0 + b_1x + b_2x^2 + b_3x^3 ... + b_mx^m,\ a_m \neq 0$

be two polynomials of degree *m* and *n* respectively in *R[x]*, then by the definition of the sum of two polynomials we have,

deg. [*f(x)* + *g(x)*] = $\begin{cases} \max.\ (m,\ n) & if\quad m \neq n \\ m \quad if \quad m = n \quad and \quad a_m + b_n \neq 0 \\ \lhd\ m \quad if \quad m = n \quad and \quad a_m + b_n = 0 \end{cases}$

Showing thereby that,

deg. [*f(x)* + *g(x)*] $\leq$ max. [*deg. f(x), deg. g(x)*].

2.5.4 Multiplication of Polynomials

Let $f(x) = a_0x^0 + a_1x + a_2x^2 + a_3x^3 \ldots + a_nx^n + \ldots$

and $g(x) = b_0x^0 + b_1x + b_2x^2 + b_3x^3 \ldots + b_nx^n + \ldots$

be any two polynomials in *R[x]*, then the product of *f(x)* and *g(x)* denote by *f(x)g(x)* is defined by

$$f(x)g(x) = c_0x^0 + c_1x + c_2x^2 + \ldots + c_ix^i + \ldots$$

where

$$c_i = \sum_{j+k=i} a_jb_k,\ \forall j = 0,1,2,\ldots$$

$$c_0 = \sum_{j+k=o} a_jb_k = a_0b_0$$

$$c_1 = \sum_{j+k=1} a_jb_k = a_0b_1 + a_1b_0$$

$$c_2 = \sum_{j+k=2} a_jb_k = a_0b_2 + a_1b_1 + a_2b_0$$

...

$$c_i = \sum_{j+k=i} a_jb_k = a_0b_1 + a_1b_{i-1} + a_2b_{i-2} + \ldots + a_ib_0$$

to obtain *f(x)g(x)* multiply the expressions $(a_0x^0 + a_1x + \ldots + a_nx^n + \ldots)$ and $(b_0x^0 + b_1x + \ldots + b_nx^n + \ldots)$ by multiplying out the symbols formally and using the relation $x^ix^j = x^{i+j}$, collect the coefficients of different powers of *x*. Clearly, the product of two polynomials is *R[x]* is again a polynomial in *R[x]*.

Also, it is obvious that for any two polynomials *f(x)* and *g(x)* in *R[x]*, deg. [*f(x) g(x)*] $\leq$ deg. *f(x)* + deg. *g(x)*, for if,

$$f(x) = a_0x^0 + a_1x + a_2x^2 + a_3x^3 ... + a_mx^m, \quad a_m \neq 0$$

and $g(x) = b_0x^0 + b_1x + b_2x^2 + b_3x^3 ... + b_nx^n, \quad a_n \neq 0$

be any two polynomials of degree *m* and *n* respectively in *R[x]*, then by the definition of the product of two polynomials we have,

$$\begin{aligned}
f(x)g(x) &= a_0b_0x^0 + a_0b_1x + a_0b_2x^2 + a_0b_3x^3 + ... + a_0b_nx^n \\
&\quad + a_1b_0x + a_1b_1x^2 + a_1b_2x^3 + a_1b_3x^4 + ... + a_1b_nx^{1+n} \\
&\quad + a_2b_0x^2 + a_2b_1x^3 + a_2b_2x^4 + a_2b_3x^5 + ... + a_2b_nx^{2+n} \\
&\quad + a_3b_0x^3 + a_3b_1x^4 + a_3b_2x^5 + a_3b_3x^6 + ... + a_3b_nx^{3+n} \\
&\quad . \\
&\quad . \\
&\quad . \\
&\quad + a_mb_0x^m + a_mb_1x^{m+1} + a_mb_2x^{m+2} + a_mb_3x^{m+3} + ... + a_mb_nx^{m+n} \\
&= (a_0b_0)x^0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 \\
&\quad + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + (a_1b_3 + a_2b_2 + a_3b_1)x^4 \\
&\quad + (a_2b_3 + a_3b_2)x^5 + (a_3b_3)x^6 + ... \\
&\quad + a_0b_nx^n + a_1b_nx^{1+n} + a_2b_nx^{2+n} + a_3b_nx^{3+n} + ... + a_mb_nx^{m+n}
\end{aligned}$$

and so if $a_mb_n \neq 0$, then

deg. [*f(x) g(x)*] = *m* + *n* = deg. *f(x)* + deg. *g(x)*.

Also if $a_mb_n = 0$, then

deg. [*f(x) g(x)*] < *m* + *n* = deg. *f(x)* + deg. *g(x)*.

Hence, deg. [*f(x) g(x)*] ≤ deg. *f(x)* + deg. *g(x)*,

In case the ring *R* is an integral domain, it is without zero divisors and therefore $a_m \neq 0, b_n \neq 0 \Rightarrow a_mb_n \neq 0$

and so in this case,

deg. [*f(x) g(x)*] = *m* + *n* = deg. *f(x)* + deg. *g(x)*

2.5.5 Example of Sum and Product of Polynomials

Example 1.

Let

$$f(x)=2x^0+3x+5x^2$$
and
$$g(x)=3x^0-5x+4x^2-9x^3$$

be any two polynomials over the ring of integers.

Then we may write

$$f(x)=2x^0+3x+5x^2+0x^3=a_0x^0+a_1x+a_2x^2+a_3x^3$$

and

$$g(x)=3x^0-5x+4x^2-9x^3=b_0x^0+b_1x+b_2x^2+b_3x^3$$

where

$$a_0=2,\ a_1=3,\ a_2=5,\ a_3=0$$

and

$$b_0=3,\ b_1=-5,\ b_2=4,\ b_3=-9$$

Now, by definition of sum of polynomials we have,

$$f(x)+g(x)=(a_0+b_0)x^0+(a_1+b_1)x+(a_2+b_2)x^2+(a_3+b_3)x^3$$

$$=(2+3)x^0+(3-5)x+(5+4)x^2+(0-9)x^3$$

$$=5x^0-2x+9x^2-9x^3$$

Also by definition of the product of polynomials we have,

$$f(x)g(x)=c_0x^0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5$$

where

$$c_0 = \sum_{j+k=0} a_j b_k = a_0 b_0 = 2\times 3 = 6$$

$$c_1 = \sum_{j+k=1} a_j b_k = a_0 b_1 + a_1 b_0 = 2\times(-5) + (3\times 3) = -10 + 9 = -1$$

$$c_2 = \sum_{j+k=2} a_j b_k = a_0 b_2 + a_1 b_1 + a_2 b_0 = (2\times 4) + 3\times(-5) + (5\times 3) = 8$$

$$c_3 = \sum_{j+k=3} a_j b_k = a_0 b_3 + a_1 b_3 + a_3 b_1 + a_3 b_0$$
$$= 2\times(-9) + (3\times 4) + 5\times(-5) + 0\times 3 = -31$$

$$c_4 = \sum_{j+k=4} a_j b_k = a_1 b_3 + a_2 b_2 + a_3 b_1 = 3\times(-9) + (5\times 4) + 0\times(-5) = -7$$

$$c_5 = \sum_{j+k=5} a_j b_k = a_2 b_3 + a_3 b_2 = 5\times(-9) + (0\times 4) = -45$$

Hence, $f(x)g(x) = 6x^0 - 1x + 8x^2 - 31x^3 - 7x^4 - 45x^5$.

Example 2.

Consider the set $I_5 = \{0, 1, 2, 3, 4\}$. It can easily verify that I_5 is a ring R with respect to addition modulo 5 and multiplication modulo 5.

Now let,

$f(x) = 3x^0 + 4x + 2x^2$

and

$g(x) = 1x^0 + 3x + 4x^2 + 2x^3$

be any two polynomials over the ring $(I_5, +_5, \times_5)$.

Then we may write

$$f(x) = 3x^0 + 4x + 2x^2 + 0x^3 = a_0x^0 + a_1x + a_2x^2 + a_3x^3$$

and

$$g(x) = 1x^0 + 3x + 4x^2 + 2x^3 = b_0x^0 + b_1x + b_2x^2 + b_3x^3$$

where

$$a_0 = 3,\ a_1 = 4,\ a_2 = 2,\ a_3 = 0$$

and

$$b_0 = 1,\ b_1 = 3,\ b_2 = 4,\ b_3 = 2$$

So by definition of sum of polynomials we have,

$$\begin{aligned} f(x) + g(x) &= (a_0 +_5 b_0)x^0 + (a_1 +_5 b_1)x + (a_2 +_5 b_2)x^2 + (a_3 +_5 b_3)x^3 \\ &= (3 +_5 1)x^0 + (4 +_5 3)x + (2 +_5 4)x^2 + (0 +_5 2)x^3 \\ &= 4x^0 + 2x + 1x^2 + 2x^3. \end{aligned}$$

Also by definition of the product of polynomials we have,

$$f(x)g(x) = c_0x^0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5$$

where,

$$c_0 = \sum_{j+k=0} a_j \times_5 b_k = a_0 \times_5 b_0 = 3 \times_5 1 = 3$$

$$c_1 = \sum_{j+k=1} a_j \times_5 b_k = (a_0 \times_5 b_1) +_5 (a_1 \times_5 b_0)$$
$$= 3 \times_5 3 +_5 (4 \times_5 1) = 4 +_5 4 = 3$$

$$c_2 = \sum_{j+k=2} a_j \times_5 b_k = (a_0 \times_5 b_2) +_5 (a_1 \times_5 b_1) +_5 (a_2 \times_5 b_0)$$
$$= (3 \times_5 4) +_5 (4 \times_5 3) +_5 (2 \times_5 1) = 1$$

$$c_3 = \sum_{j+k=3} a_j \times_5 b_k = (a_0 \times_5 b_3) +_5 (a_1 \times_5 b_3) +_5 (a_3 \times_5 b_1) +_5 (a_3 \times_5 b_0)$$
$$= (3 \times_5 2) +_5 (4 \times_5 4) +_5 (2 \times_5 3) +_5 (0 \times_5 1) = 3$$

$$c_4 = \sum_{j+k=4} a_j \times_5 b_k = (a_1 \times_5 b_3) +_5 (a_2 \times_5 b_2) +_5 (a_3 \times_5 b_1)$$
$$= (4 \times_5 2) +_5 (2 \times_5 4) +_5 (0 \times_5 1) = 1$$

$$c_5 = \sum_{j+k=5} a_j \times_5 b_k = (a_2 \times_5 b_3) +_5 (a_3 \times_5 b_2) = (2 \times_5 2) +_5 (0 \times_5 4) = 4$$

Hence, $f(x)g(x) = 3x^0 + 3x + 1x^2 + 3x^3 + 1x^4 + 4x^5$.

2.6 Near Ring

The meaning or definition of near ring is not more different with ring, because the axioms of near ring is almost same with ring, such that near ring can be called as a developing of ring.

Definition. A set of G with two binary operations (additive "+" and multiplication), $G \neq \varnothing$. $(G, +, \times)$ is called near ring if following these axioms:

1) (G, +) is a group

2) $(G, \times)$ is semigroup

3) Multiplication is right distributive of additive operation, such that

$\forall a, b, c \in R$, *such that* $(a+b)\times c = (a\times c)+(b\times c)$

(Mayr, 2000: 185).

We know from definition above there is difference among near ring and ring, such ring, additive operation at group must be abelian. While at near ring, additive operation at group must not be abelian. Hence, ring is clearly near ring, but not contradiction.

Example. Set of integer Z as additive operation and multiplication, Is $(Z, +, \times)$ near ring?

Answered

1) (Z, +) is a group if

a. Z is closed at additive operation

Choose any $a, b \in Z$, *so that* $a+b \in Z$

b. Additive operation of Z is associative

Choose any $a, b, c \in Z$, *so that* $a+(b+c)=(a+b)+c$

c. $\forall a \in Z, \exists\, 0 \in Z$, *so that* $a+0=0+a=a$

d. For every element in Z has inverse

$\forall a \in Z, \exists -a \in Z$, *so that* $a+(-a)=(-a)+a=0$

Hence, (Z, +) is a group.

2) $(Z, \times)$ is semigroup if

a) Z is closed at multiplication operation

Choose any $a, b \in Z$, $so\,that$ $a \times b \in Z$

b) Multiplication operation of Z is associative

Choose any $a, b, c \in Z$, $so\,that$ $a \times (b \times c) = (a \times b) \times c$

3) Multiplication is right distributive of additive operation, such that

$\forall a, b, c \in R$, *such that* $(a + b) \times c = (a \times c) + (b \times c)$

Hence, $(Z, +, \times)$ is near ring.

2.7 Ring Studies in Religion

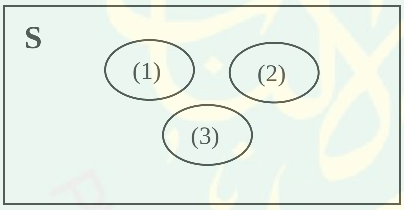
In general, some concepts from the disciplines of science have been described in the Qur'an, one of which is mathematics. The concept of mathematical disciplines that exist in Qur'an is the problem of logic, modeling, statistics, graph theory, the theory of groups and others. Theories about the group, where the definition of the group itself an algebraic structure which is expressed as $(G,*)$, with G does not equal the empty set $(G \neq \emptyset)$ and $*$ are binary operations on G, which are assosiative, there is an identity, and there was inverse in the group. Similar sets of graph theory in the group or members who have elements also a creature of his creation. While the binary operation is an interaction between His creatures and the properties that must be met are the rules that have been established by Allah means that even creatures interact with fellow creature he must stay within a corridor that has been ordained by Allah.

Set studies exists in the Qur'an. For example, human life consists of various groups. The group is also a set, because the set itself are collection of objects defined. In Qur’an surah Al-Fatihah verse 7 is mentioned.

صِرَٰطَ ٱلَّذِينَ أَنۡعَمۡتَ عَلَيۡهِمۡ غَيۡرِ ٱلۡمَغۡضُوبِ عَلَيۡهِمۡ وَلَا ٱلضَّآلِّينَ ٧

It means: “The part of those whom Thou hast favoured” (Surah Al-Fatihah: 7)

The meaning of the verse is that humans are divided into three groups, namely (1) group which received blessings from Allah SWT (2) groups that accursed (3) a misguided group (Abdusysyakir, 2006: 47) like the following picture.

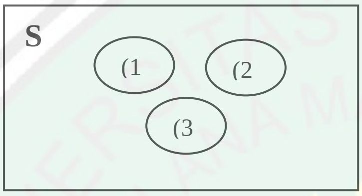


Talking about the set of humans, is also mentioned in the Qur'an sets the other. Notice the word Allah in surah Al-Faathir verse 1.

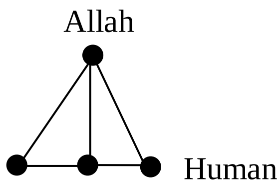
ٱلۡحَمۡدُ لِلَّهِ فَاطِرِ ٱلسَّمَٰوَٰتِ وَٱلۡأَرۡضِ جَاعِلِ ٱلۡمَلَٰٓئِكَةِ رُسُلًا أُوْلِيٓ أَجۡنِحَةٖ
مَّثۡنَىٰ وَثُلَٰثَ وَرُبَٰعَۚ يَزِيدُ فِي ٱلۡخَلۡقِ مَا يَشَآءُۚ إِنَّ ٱللَّهَ عَلَىٰ كُلِّ شَيۡءٖ قَدِيرٞ ١

It means: “Praise be to Allah, the Creator of the heavens and the earth, who appointeth the angels as messengers having wings two, three and four. He multiplieth in certain what He will. Lo! Allah is Able to do all things” (Surah Al-Faathir: 1)

In verse 1 of this surah Al-Faathir described a group, party or group of beings called angels. In groups of angels there are groups of angels that have two wings, three wings, or four wings. Even very possible there is a group of angels who have more than four wings if Allah willed (Abdussakir, 2006: 48). As the following figure:



Back on definition of the group that is non-empty set with binary operations that satisfy the properties assosiative, no identity, and no inverse. After discussing the Islamic concept of set, now studying in the Islamic concept of binary operation.In example $\circ$ is the operation on elements of S then it is called a binary operation, if any two elements $a, b \in S$ then $(a \circ b) \in S$. So if a member of the set S is operated result also a member of$S$. In the real world of binary operations and attributes that must be met by a group between fellow creatures. So even though these creatures interact with a variety of pattern will remain in the set that is the set of his creation, as in the picture below.

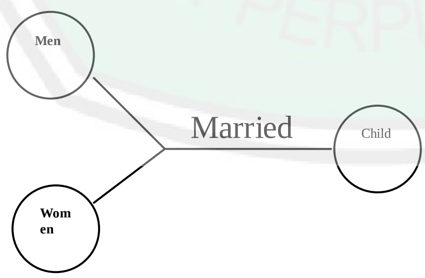


Algebraic system is one of the materials on the part of abstract algebra that contains a binary operation. Association with one or more binary operations called algebraic systems. Algebraic system with binary operations that satisfy certain properties are closed, assosiative, inverse, identity, later known group. While studies with a set of binary operations in the Islamic concept that is, that man is created in pairs. While the study group in the Islamic concept namely, that human beings are created in pairs. Notice the word Allah in surah Al-Faathir verse 11.

وَٱللَّهُ خَلَقَكُم مِّن تُرَابٖ ثُمَّ مِن نُّطۡفَةٖ ثُمَّ جَعَلَكُمۡ أَزۡوَٰجٗاۚ وَمَا تَحۡمِلُ مِنۡ أُنثَىٰ
وَلَا تَضَعُ إِلَّا بِعِلۡمِهِۦۚ وَمَا يُعَمَّرُ مِن مُّعَمَّرٖ وَلَا يُنقَصُ مِنۡ عُمُرِهِۦٓ إِلَّا فِي
كِتَٰبٍۚ إِنَّ ذَٰلِكَ عَلَى ٱللَّهِ يَسِيرٞ ١١

It means: "Allah created you from dust, then from a little fluid, the He made you pairs (the male and famale). No female beareth or bringeth forth save with His knowladge. And no one groweth old, nor is aught lessened of his life, but it is recorded in a Book. Lo! That is easy for Allah" (Surah Al-Faathir: 11)

From the word of the above that our people are in pairs of men with women, so men and women must be in pairs, and the paired (married) man can conceive and give birth to a child and then the child will also be paired with another child, such as the following picture



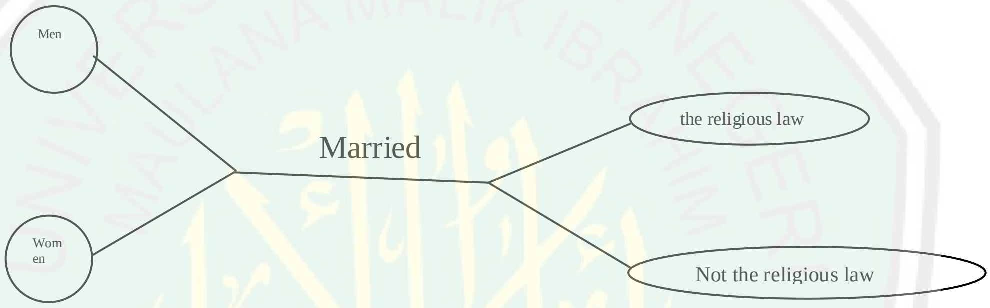
Or (M, N) with M is the set of human {male, female} and N is the wedding.

While the definition of the ring is suppose R is not an empty set with two binary operations $+$ and $\cdot$ (called the sum / first operation and the multiplication/ second operation) is called a ring if it meets the following statement: $(R, +)$ is Abelian groups, operation $\cdot$ assosiative, operation $\cdot$ distributive of the operation $+$. For example $\circ$ and $\bullet$ is the operation to the elements S and he is called binary if every two elements $a, b \in S$ then $(a \circ b) \in S$ and $(a \bullet b) \in S$. So if the member of set S operating resulted were also members of S. If linked to the Islamic concept, consider the word of Allah in surah Al-Nisaa verse 23

حُرِّمَتۡ عَلَيۡكُمۡ أُمَّهَٰتُكُمۡ وَبَنَاتُكُمۡ وَأَخَوَٰتُكُمۡ وَعَمَّٰتُكُمۡ وَخَٰلَٰتُكُمۡ وَبَنَاتُ
ٱلۡأَخِ وَبَنَاتُ ٱلۡأُخۡتِ وَأُمَّهَٰتُكُمُ ٱلَّٰتِيٓ أَرۡضَعۡنَكُمۡ وَأَخَوَٰتُكُم مِّنَ
ٱلرَّضَٰعَةِ وَأُمَّهَٰتُ نِسَآئِكُمۡ وَرَبَٰٓئِبُكُمُ ٱلَّٰتِي فِي حُجُورِكُم مِّن نِّسَآئِكُمُ
ٱلَّٰتِي دَخَلۡتُم بِهِنَّ فَإِن لَّمۡ تَكُونُواْ دَخَلۡتُم بِهِنَّ فَلَا جُنَاحَ عَلَيۡكُمۡ
وَحَلَٰٓئِلُ أَبۡنَآئِكُمُ ٱلَّذِينَ مِنۡ أَصۡلَٰبِكُمۡ وَأَن تَجۡمَعُواْ بَيۡنَ ٱلۡأُخۡتَيۡنِ إِلَّا
مَا قَدۡ سَلَفَۗ إِنَّ ٱللَّهَ كَانَ غَفُورٗا رَّحِيمٗا ٢٣

It means:"Forbidden unto you are your mothers, and you daughters and your sisters and your father's sisters and your mother's sisters and your brother's daughters and your sister's daughters and your foster-mothers and your foster-sisters and your mothers-inlaw and your step-daughters who are under your protection (born) of your women unto whom ye have gone in---but if ye have not gone in unto them, the it is no sin for you (to marry their daughters)— and the wives of your sons who (spring) form your own loins. And (it is forbidden unto you) that ye should have two sisters together, except what hath already happend (of that nature) in the past. Lo! Allah is never forgiving merciful." (surah Al-Nisaa: 23)

So from the word above our people are in pairs between men and women by marriage. But how to get married with their spouses, should be legally religion and if it is not in accordance with religious law, it is forbidden for both partners to be married. Whereas the purpose of marriage to be lawful. So you get married with spouse in accordance with religious law, such as the following figure.



Or (M, N, H) and M is the set of human $\{$male, female$\}$, N is the wedding and H is the religious law.

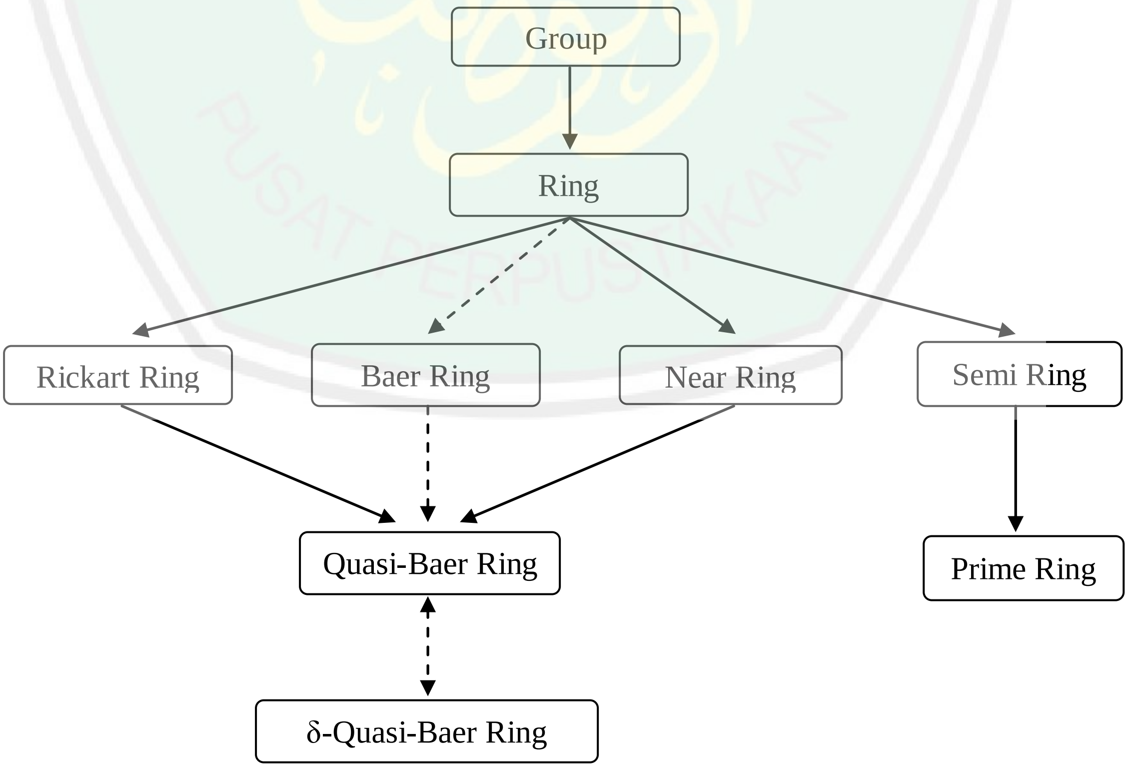
CHAPTER III

DISCUSSION

3.1 Quasi-Baer Ring

Throughout this thesis all rings are associative and all nearrings are left nearrings. We use R and N to denote a ring and nearring respectively. The study of *Rickart rings* has its roots in both functional analysis and homological algebra. Rickart studies C*-algebra with the property that every right annihilator of any element is generated by idempotent. This condition is modified by *Kaplansky* through introducing *Baer ring* to abstract various properties of AW*-algebra and Von Neumann algebra.

Flowchart The History of δ-Quasi-Baer Ring :



A ring satisfying a generalization of Rickart's condition (every right annihilator of any element in *R*, as a right ideal, by an idempotent) has homological characterization as a right *PP* ring. Left *PP* ring are defined similarly. Then *Clark* used the quasi-baer concept to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Every prime ring is a quasi-baer ring. It is natural to ask if some of these properties can extended from a ring *R* to the polynomial ring.

Definition. Let *R* is a ***Baer ring*** if the right annihilator of every nonempty subset of *R* is generated by idempotent (Kaplansky, 1965: 78).

Explaining of idempotent is same with meaning of projection. An idempotent *p* is called a projection if $p = p^*$ (where * is an involution on the algebra).

In (Clark, 1967: 417-424) Clark defines a ring to be ***Quasi-Baer*** if the *right* annihilator of every ideal (as a *right* ideal) is generated by idempotent. Moreover, he shows the left-right symmetry of this condition by proving that *R* is Quasi-Baer if and only if the *left* annihilator of every ideal (as a *left* ideal) is generated by idempotent.

3.1.1 Annihilator

Definition

Let for a non-empty subset (any elements) *X* of a ring *R*, we write $r_R(X) = \{c \in R \mid dc = 0, \forall d \in X\}$ is called the *right annihilator* of *X* in *R* (Hashemi, 2007: 197-200).

Definition

Let for a non-empty subset (any elements) X of a ring R, we write $l_R(X) = \{c \in R \mid cd = 0,\ \forall d \in X\}$ is called the *left annihilator* of X in R (Hashemi, 2007: 197-200).

According to the writer from these definition (annihilator) above, the writer explain that there is at least one of element in $r_R(X)$ or $l_R(X)$ is I (identity).

Looking at $r_R(X) = \{c \in R \mid dc = 0,\ \forall d \in X\}$ is right annihilator. Let 0 is right annihilator if there is $0 \in R \rightarrow d \times 0 = 0$; $\forall d \in X$, therefore $X \subseteq R$.

3.1.2 Example. Let $(M_8, +, \times)$ be a ring R of arbitrary integer set in modulo 8. Show that from M_8 is Quasi-Baer ring. If we consider the set $M_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$, then M_8 is a ring with respect to addition modulo 8 and multiplication modulo 8. Here *0* is the identity for $+_8$.

Answered. $M_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$

$\bar{0} = \bar{8} = \overline{16} = \overline{24} = \overline{32} = \overline{40} = \overline{48} = \overline{56} = \overline{64}$

Let p is idempotent, such that

$\rightarrow p^2 = 0$

$p \times p = 0$

$\bar{0}$ is idempotent because $\exists\ \bar{0} \ni \bar{0} \times \bar{0} = \bar{0}$

$\bar{1}$ is not idempotent because $\exists\ \bar{1} \ni \bar{1} \times \bar{1} = \bar{1} \neq \bar{0}$

$\bar{2}$ is not idempotent because $\exists\ \bar{2} \ni \bar{2}\times\bar{2}=\bar{4}\neq\bar{0}$

$\bar{3}$ is not idempotent because $\exists\ \bar{3} \ni \bar{3}\times\bar{3}=\bar{1}\neq\bar{0}$

$\bar{4}$ is idempotent because $\exists\ \bar{4} \ni \bar{4}\times\bar{4}=\bar{8}=\bar{0}$

$\bar{5}$ is not idempotent because $\exists\ \bar{5} \ni \bar{5}\times\bar{5}=\bar{1}\neq\bar{0}$

$\bar{6}$ is not idempotent because $\exists\ \bar{6} \ni \bar{6}\times\bar{6}=\bar{4}\neq\bar{0}$

$\bar{7}$ is not idempotent because $\exists\ \bar{7} \ni \bar{7}\times\bar{7}=\bar{1}\neq\bar{0}$

Hence, idempotent of M_8 is $\bar{0}$ and $\bar{4}$.

Now, Looking at $r_R(X)=\{c\in R \mid dc=0,\ \forall d\in X\}$ is right annihilator.

Let $A\subseteq M_8$, $A=\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$. There is $r\in M_8 \to r\times A=\bar{0}$ is called right annihilator, such that

$\bar{0}\times\bar{0}=\bar{0}$
$\bar{0}\times\bar{2}=\bar{0}$
$\bar{0}\times\bar{4}=\bar{0}$
$\bar{0}\times\bar{6}=\bar{0}$
Therefore, $\bar{0}\in r_R(X)$

$\bar{1}\times\bar{0}=\bar{0}$
$\bar{1}\times\bar{2}=\bar{2}\neq\bar{0}$
$\bar{1}\times\bar{4}=\bar{4}\neq\bar{0}$
$\bar{1}\times\bar{6}=\bar{6}\neq\bar{0}$
Therefore, $\bar{1}\notin r_R(X)$

$$\bar{2}\times\bar{0}=\bar{0}$$
$$\bar{2}\times\bar{2}=\bar{4}\neq\bar{0}$$
$$\bar{2}\times\bar{4}=\bar{8}=\bar{0}$$
$$\bar{2}\times\bar{6}=\overline{12}=\bar{4}\neq\bar{0}$$
$$Therefore,\ \bar{2}\notin r_R(X)$$

$$\bar{3}\times\bar{0}=\bar{0}$$
$$\bar{3}\times\bar{2}=\bar{6}\neq\bar{0}$$
$$\bar{3}\times\bar{4}=\overline{12}=\bar{4}\neq\bar{0}$$
$$\bar{3}\times\bar{6}=\overline{18}=\bar{2}\neq\bar{0}$$
$$Therefore,\ \bar{3}\notin r_R(X)$$

$$\bar{4}\times\bar{0}=\bar{0}$$
$$\bar{4}\times\bar{2}=\bar{8}=\bar{0}$$
$$\bar{4}\times\bar{4}=\overline{16}=\bar{0}$$
$$\bar{4}\times\bar{6}=\overline{24}=\bar{0}$$
$$Therefore,\ \bar{4}\in r_R(X)$$

$$\bar{5}\times\bar{0}=\bar{0}$$
$$\bar{5}\times\bar{2}=\overline{10}=\bar{2}\neq\bar{0}$$
$$\bar{5}\times\bar{4}=\overline{20}=\bar{4}\neq\bar{0}$$
$$\bar{5}\times\bar{6}=\overline{30}=\bar{6}\neq\bar{0}$$
$$Therefore,\ \bar{5}\notin r_R(X)$$

$$\bar{6}\times\bar{0}=\bar{0}$$
$$\bar{6}\times\bar{2}=\overline{12}=\bar{4}\neq\bar{0}$$
$$\bar{6}\times\bar{4}=\overline{24}=\bar{0}$$
$$\bar{6}\times\bar{6}=\overline{36}=\bar{4}\neq\bar{0}$$
$$Therefore,\ \bar{6}\notin r_R(X)$$

$\overline{7} \times \overline{0} = \overline{0}$

$\overline{7} \times \overline{2} = \overline{14} = \overline{6} \neq \overline{0}$

$\overline{7} \times \overline{4} = \overline{28} = \overline{4} \neq \overline{0}$

$\overline{7} \times \overline{6} = \overline{42} = \overline{2} \neq \overline{0}$

Therefore, $\overline{7} \notin r_R(X)$

Hence, right annihilator / $r_R(X)$ of M_8 is $\overline{0}$ and $\overline{4}$.

As we know that idempotent of M_8 is $\overline{0}$ and $\overline{4}$ and right annihilator / $r_R(X)$ of M_8 is $\overline{0}$ and $\overline{4}$, so that M_8 is Quasi-Baer ring.

3.1.3 A Derivation of R (δ)

Definition. Let $\delta : R \rightarrow R$ is derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \delta a(b),\ \forall\, a, b \in R$.

Definition. Let *R[x; δ]* is the polynomial ring whose elements are the polynomials denote $\sum_{i=0}^{n} r_i x^i \in R,\ r_i \in R$, where the multiplication by $xb = bx + \delta(b),\ \forall\ b \in R$

(Hashemi, 2007: 197-200).

3.2 δ-Quasi-Baer Ring

Definition. Let δ be a derivation on *R*. A ring *R* is called δ-Quasi-Baer ring if the right annihilator ($r_R(X) = \{c \in R \mid dc = 0,\ \forall\, d \in X\}$) of every δ-ideal of *R* is generated by idempotent.

An ideal I of R is called δ-*ideal* if $\delta(I) \subseteq I$. R is called δ-Quasi-Baer ring if the right annihilator of every δ-ideal of R is generated by idempotent of R.

In (Armendariz, 1974: 470-473) Armendariz has shown that if R is reduced, then R is Baer if and only if the polynomial ring *R[x]* is a Baer ring. In (J. Han, Y. Hirano and H. Kim, 2000) have generalized the explaining of δ-Quasi-Baer by showing that if R is *δ-semiprime* (for any δ-ideal I of R, $I^2 = 0$ implies $I = 0$ is called idempotent I), then R is a δ-Quasi-Baer ring if and only if the extension *R[x; δ]* is a Quasi-Baer ring.

Example. Let $(Z_7, \oplus, \delta)$ be a ring R of arbitrary integer set in modulo 7. Show that from Z_7 is δ-Quasi-Baer ring. If we consider the set $Z_7 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$, then Z_7 is a ring with respect to "$\oplus$" modulo 7 and "δ" modulo 7. Here *0* is the identity for "$\oplus_7$".

Answered. $Z_7 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$

$\bar{0} = \bar{7} = \overline{14} = \overline{21} = \overline{28} = \overline{35} = \overline{42} = \overline{49} = \overline{56}$

Let p is idempotent, such that

$\rightarrow p^2 = 0$

$p \times p = 0$

$\bar{0}$ is idempotent because $\exists\, \bar{0} \ni \bar{0}\delta\bar{0} = \bar{0}$

$\bar{1}$ is not idempotent because $\exists\, \bar{1} \ni \bar{1}\delta\,\bar{1} \neq \bar{0}$

$\bar{2}$ is not idempotent because $\exists\, \bar{2} \ni \bar{2}\delta\bar{2} \neq \bar{0}$

$\bar{3}$ is not idempotent because $\exists\, \bar{3} \ni \bar{3}\delta\bar{3} \neq \bar{0}$

$\bar{4}$ is not idempotent because $\exists\, \bar{4} \ni \bar{4}\delta\bar{4} \neq \bar{0}$

$\bar{5}$ is not idempotent because $\exists\, \bar{5} \ni \bar{5}\delta\bar{5} \neq \bar{0}$

$\bar{6}$ is not idempotent because $\exists\, \bar{6} \ni \bar{6}\delta\bar{6} \neq \bar{0}$

Hence, idempotent of Z_7 is $\bar{0}$.

Now, Looking at $r_R(X) = \{c \in R \mid dc = 0,\ \forall\, d \in X\}$ is right annihilator.

Let $B \subseteq Z_7,\ B = \{\bar{0}, \bar{1}, \bar{3}, \bar{5}\}$. There is $r \in Z_7 \to r\,\delta\, B = \bar{0}$ is called right annihilator, such that

$\bar{0}\delta\bar{0} = \bar{0}$
$\bar{0}\delta\bar{1} = \bar{0}$
$\bar{0}\delta\bar{3} = \bar{0}$
$\bar{0}\delta\bar{5} = \bar{0}$
Therefore, $\bar{0} \in r_R(X)$

$\bar{1}\delta\bar{0} = \bar{0}$
$\bar{1}\delta\bar{1} \neq \bar{0}$
$\bar{1}\delta\bar{3} \neq \bar{0}$
$\bar{1}\delta\bar{5} \neq \bar{0}$
Therefore, $\bar{1} \notin r_R(X)$

$\bar{2}\delta\bar{0} = \bar{0}$
$\bar{2}\delta\bar{1} \neq \bar{0}$
$\bar{2}\delta\bar{3} \neq \bar{0}$
$\bar{2}\delta\bar{5} \neq \bar{0}$
Therefore, $\bar{2} \notin r_R(X)$

$\bar{3}\delta\bar{0}=\bar{0}$
$\bar{3}\delta\bar{1}\neq\bar{0}$
$\bar{3}\delta\bar{3}\neq\bar{0}$
$\bar{3}\delta\bar{5}\neq\bar{0}$
$Therefore,\ \bar{3}\notin r_R(X)$

$\bar{4}\delta\bar{0}=\bar{0}$
$\bar{4}\delta\bar{1}\neq\bar{0}$
$\bar{4}\delta\bar{3}\neq\bar{0}$
$\bar{4}\delta\bar{5}\neq\bar{0}$
$Therefore,\ \bar{4}\notin r_R(X)$

$\bar{5}\delta\bar{0}=\bar{0}$
$\bar{5}\delta\bar{1}\neq\bar{0}$
$\bar{5}\delta\bar{3}\neq\bar{0}$
$\bar{5}\delta\bar{5}\neq\bar{0}$
$Therefore,\ \bar{5}\notin r_R(X)$

$\bar{6}\delta\bar{0}=\bar{0}$
$\bar{6}\delta\bar{1}\neq\bar{0}$
$\bar{6}\delta\bar{3}\neq\bar{0}$
$\bar{6}\delta\bar{5}\neq\bar{0}$
$Therefore,\ \bar{6}\notin r_R(X)$

Hence, right annihilator $/\,r_R(X)$ of Z_7 is $\bar{0}$.

As we know that idempotent of Z_7 is $\bar{0}$ and right annihilator $/\,r_R(X)$ of Z_7 is $\bar{0}$, so that Z_7 is δ-Quasi-Baer ring.

3.3 Characterization of δ-Quasi-Baer Ring

In algebra, we know that if there is $z = x + yi$ and the its conjugate is $\bar{z} = x - yi$, it mean that $\bar{z}$ is an extention of z. Similarly, if there is $\delta : R \to R$ is derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \delta a(b), \forall a,b \in R$, so that the extension of $\delta : R \to R$ is $\bar{\delta}(ab) = \delta(a)b - \delta a(b), \forall a,b \in R$.

Definition. Let R be a Ring. Then an infinite ordered set $(a_0, a_1, \ldots, a_n \ldots)$ of elements of R with at most a finite number of non-zero elements is called a polynomial over a ring R (Raisinghania, 1980: 422).

In other words an infinite sequence $(a_0, a_1, \ldots, a_n, \ldots)$ of elements of R is said to be a polynomial over R if there exists a non-negative integer n such that $a_i = 0$, the zero of the ring R $\forall$ $i > n$. We denote such a polynomial by the symbol

$$f(x) = a_0 x^0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + 0x^{n+1} + 0x^{n+2} \ldots = \sum_{i=0}^{\infty} a_i x^i$$

Where the symbol x is not an element of R and is known as an indeterminate over the ring R. We call $a_0 x^0, a_1 x^1, a_2 x^2, \ldots, a_n x^n, \ldots$ etc, as the terms of this polynomial *f(x)* and the elements $(a_0, a_1, \ldots, a_n, \ldots)$ of R as the coefficients of these terms. The different powers of x simply indicate the ordered place of the different coefficients and the symbol '+' in between the different terms indicates the separation of the terms.

Example. Consider the set I_5 = {0, 1, 2, 3, 4}. It can easily verify that I_5 is a ring R with respect to addition modulo 5 and multiplication modulo 5.

Now let,

$$f(x) = 3x^0 + 4x + 2x^2$$
and
$$g(x) = 1x^0 + 3x + 4x^2 + 2x^3$$

be any two polynomials over the ring $(I_5, +_5, \times_5)$.

Then we may write

$$f(x) = 3x^0 + 4x + 2x^2 + 0x^3 = a_0x^0 + a_1x + a_2x^2 + a_3x^3$$

and

$$g(x) = 1x^0 + 3x + 4x^2 + 2x^3 = b_0x^0 + b_1x + b_2x^2 + b_3x^3$$

where

$$a_0 = 3,\ a_1 = 4,\ a_2 = 2,\ a_3 = 0$$

and

$$b_0 = 1,\ b_1 = 3,\ b_2 = 4,\ b_3 = 2$$

So by definition of sum of polynomials we have,

$$f(x) + g(x) = (a_0 +_5 b_0)x^0 + (a_1 +_5 b_1)x + (a_2 +_5 b_2)x^2 + (a_3 +_5 b_3)x^3$$

$$= (3 +_5 1)x^0 + (4 +_5 3)x + (2 +_5 4)x^2 + (0 +_5 2)x^3$$

$$= 4x^0 + 2x + 1x^2 + 2x^3.$$

Also by definition of the product of polynomials we have,

$$f(x)g(x) = c_0x^0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5$$

where,

$$c_0 = \sum_{j+k=0} a_j \times_5 b_k = a_0 \times_5 b_0 = 3 \times_5 1 = 3$$

$$c_1 = \sum_{j+k=1} a_j \times_5 b_k = (a_0 \times_5 b_1) +_5 (a_1 \times_5 b_0)$$
$$= 3 \times_5 3 +_5 (4 \times_5 1) = 4 +_5 4 = 3$$

$$c_2 = \sum_{j+k=2} a_j \times_5 b_k = (a_0 \times_5 b_2) +_5 (a_1 \times_5 b_1) +_5 (a_2 \times_5 b_0)$$
$$= (3 \times_5 4) +_5 (4 \times_5 3) +_5 (2 \times_5 1) = 1$$

$$c_3 = \sum_{j+k=3} a_j \times_5 b_k = (a_0 \times_5 b_3) +_5 (a_1 \times_5 b_3) +_5 (a_3 \times_5 b_1) +_5 (a_3 \times_5 b_0)$$
$$= (3 \times_5 2) +_5 (4 \times_5 4) +_5 (2 \times_5 3) +_5 (0 \times_5 1) = 3$$

$$c_4 = \sum_{j+k=4} a_j \times_5 b_k = (a_1 \times_5 b_3) +_5 (a_2 \times_5 b_2) +_5 (a_3 \times_5 b_1)$$
$$= (4 \times_5 2) +_5 (2 \times_5 4) +_5 (0 \times_5 1) = 1$$

$$c_5 = \sum_{j+k=5} a_j \times_5 b_k = (a_2 \times_5 b_3) +_5 (a_3 \times_5 b_2) = (2 \times_5 2) +_5 (0 \times_5 4) = 4$$

Hence, $f(x)g(x) = 3x^0 + 3x + 1x^2 + 3x^3 + 1x^4 + 4x^5$.

For a ring R with a derivation δ, there exists a derivation on $S = R[x; \delta]$ which extends δ. Considering an inner derivation $\bar{\delta}$ on S by x defined by $\bar{\delta}(f(x)) = x\, f(x) - f(x)\, x,\ \forall\, f(x) \in S$.

Then,

$$\bar{\delta}(f(x)) = \delta(a_0) + \ldots + \delta(a_n)x^n \ \textit{for all}\ f(x) = a_0 + \ldots + a_nx^n \in S$$

$$\textit{and}\ \bar{\delta}(r) = \delta(r), \forall r \in R$$

which means that $\overline{\delta}$ is an extension of δ. We call such a derivation $\overline{\delta}$ on S an extended derivation of δ. For each $a \in R$ and nonnegative integer n, there exist $t_0, \ldots, t_n \in Z$ such that $x^n a = \sum_{i=0}^{n} t_i \delta^{n-i}(a) x^i$.

3.3.1 Theorem. Let I be a δ-*ideal* of R and $S = R[x;\, \delta]$. If $r_S(I[x;\delta]) = e(x)S$ for some idempotent $e(x) = e_0 + e_1 x + e_2 x_2 + \ldots + e_n x^n \in S$ *then* $r_S(I[x;\delta]) = e_0 S$. Therefore, $e(x)S = e_0 S$.

Proof. Since $Ie(x) = 0$, we have $Ie_i = 0$ for each $i = 0, \ldots, n$.

Hence,

$0 = \delta(Ie_i) = \delta(I)e_i + I\delta(e_i)$ *for* $i = 0, \ldots, n$. (∵ *is obtained by definition* δ)

Since I is δ-*ideal* and $Ie_i = 0$,

so that $I\delta(e_i) = 0$ for each $i = 0, \ldots, n$.

Similarly, we can show that $I\,\delta^k(e_i) = 0$ for each $i = 0, \ldots, n$ and $k \geq 0$.

Hence,

$\delta^k(e_i) \in r_S(I[x;\delta])$ for each $i = 0, \ldots, n$ and $k \geq 0$.

Thus, $\delta^k(e_i) = e(x)\,\delta^k(e_i)$

and that

$e_n \delta^k(e_i) = 0$ for each $i = 0, \ldots, n$ and $k \geq 0$.

Hence,

$\delta^k(e_i) = (e_0 + e_1 x + \ldots + e_{n-1} x^{n-1})\,\delta^k(e_i)$

and that

$e_{n-1}\,\delta^k(e_i) = 0$ for each $i \geq 0$, $k \geq 0$.

Continuing in this way, we have $e_j\delta^k(e_i) = 0$ for each $i \geq 0, k \geq 0, j = 1, \ldots, n$.

Thus, $\delta^k(e_i) = e_0\,\delta^k(e_i)$ for each $i \geq 0, k \geq 0$.

Therefore, $e(x) = e_0 e(x)$

and that $r_S(I[x;\delta]) = e(x)S \subseteq e_0 S$. ...... (i)

Since $\delta^k(e_0) \in r_R(I)$, *so* $e_0 \in r_S(I[x;\delta])$

and that $e_0 S \subseteq r_S(I[x;\delta])$. ..... (ii)

Therefore, from equations (i) and (ii) we get $r_S(I[x;\delta]) = e_0 S$.

3.3.2 Proposition. Let R be a δ-Quasi-Baer ring. Then $S = R[x;\delta]$ is a Quasi-Baer ring.

Proof. Let J be an arbitrary ideal of S. Consider the set J_0 of leading or the first coefficients of polynomials in J. Then J_0 is a δ-*ideal* of R. Since R is δ-Quasi-Baer ring, $r_R(J_0) = eR$ for some idempotent $e \in R$.

Since $J_0 e = 0$ and J_0 is δ-*ideal* of R (from J_0 is a δ-*ideal* of R and $r_R(J_0) = eR$) we have $J_0\delta^k(e) = 0$ for each $k \geq 0$.

Hence $\delta^k(e) = e\,\delta^k(e)$ and $eS \subseteq r_S(J_0[x;\delta])$.

Clearly $r_S(J_0[x;\delta]) \subseteq eS$.

Thus $r_S(J_0[x;\delta]) = eS$. ($\because$ from $eS \subseteq r_S(J_0[x;\delta])$ and $r_S(J_0[x;\delta]) \subseteq eS$)

We claim that $r_S(J) = eS$.

Let $f(x) = a_0 + \ldots + a_n x^n \in J$.

Then $a_n \in J_0$

and that $a_n\delta^k(e) = 0$ for each $k \geq 0$.

Hence $f(x)e = (a_0 + ... + a_{n-1}x^{n-1})e = ... + a_{n-1}ex^{n-1}$.

Thus $a_{n-1}e \in J_0$, and $a_{n-1}\delta^k(e) = a_{n-1}e\delta^k(e) = 0$ for each $k \geq 0$.

Hence $a_{n-1}x^{n-1}e = 0$.

Continuing in this way, we can show that $a_i x^i e = 0$ for each $i = 0, ..., n$.

Hence $f(x)e = 0$ and so $eS \subseteq r_S(J)$.

Now, let $g(x) = b_0 + ... + b_m x^m \in r_S(J)$ and $f(x) = a_0 + ... + a_n x^n \in J$.

First, we will show that $a_i x^i b_j x^j = 0$ for $i = 0, ..., n$ and $j = 0, ..., m$.

Since $f(x)g(x) = 0$, we have $a_n b_m = 0$.

Hence $b_m \in r_R(J_0)$.

Since J_0 is a δ-ideal of R, $\delta^k(b_m) \in J_0$ for each $k \geq 0$

and that $b_m \in r_S(J_0[x;\delta])$.

Thus $b_m = eb_m$ and $a_n x^n b_m x^m = 0$.

Since $f(x)e = (a_0 + ... + a_n x^n)e = (a_0 + ... + a_{n-1}x^{n-1})e$, we have $a_{n-1}e \in J_0$

and $a_{n-1}\delta^k(e) = a_{n-1}e\delta^k(e) = 0$ for each $k \geq 0$.

There exist,

$t_0, ..., t_{n-1} \in Z$ such that, $a_{n-1}x^{n-1}b_m x^m = a_{n-1}x^{n-1}eb_m x^m =$

$$a_{n-1}\left(\sum_{j=0}^{n-1} t_j \delta^{n-1-j}(e)x^j\right)b_m x^m = \left(\sum_{j=0}^{n-1} t_j a_{n-1}\delta^{n-1-j}(e)x^j\right)b_m x^m.$$

Hence $a_{n-1}x^{n-1}b_m x^m = 0$.

Continuing in this way, we have $a_i x^i b_j x^j = 0$ for each i, j.

Therefore, $b_j \in r_S(J_0[x;\delta]) = eS$, for each $j \geq 0$.

Consequently, $g(x) = eg(x)$ and $r_S(J) = eS$.

Therefore S is a Quasi-Baer ring.

3.3.3 Proposition. If R be Quasi-Baer ring with $S = R[x;\ \delta]$, such that is δ-Quasi-Baer ring.

Proof. Therefore S is a Quasi-Baer ring. Let J be an arbitrary ideal of S.

Consequently, $g(x) = eg(x)$ and $r_S(J) = eS$.

Therefore, $b_j \in r_S(J_0[x;\delta]) = eS$, for each $j \geq 0$.

Continuing in this way, we have $a_i x^i b_j x^j = 0$ for each i, j.

There exist,

$t_0, \ldots, t_{n-1} \in Z$ *such that*, $a_{n-1}x^{n-1}b_m x^m = a_{n-1}x^{n-1}eb_m x^m =$

$$a_{n-1}(\sum_{j=0}^{n-1} t_j \delta^{n-1-j}(e)x^j)b_m x^m = (\sum_{j=0}^{n-1} t_j a_{n-1}\delta^{n-1-j}(e)x^j)b_m x^m.$$

Hence $a_{n-1}x^{n-1}b_m x^m = 0$.

Since $f(x)e = (a_0 + \ldots + a_n x^n)e = (a_0 + \ldots + a_{n-1}x^{n-1})e$, we have $a_{n-1}e \in J_0$

and $a_{n-1}\delta^k(e) = a_{n-1}e\delta^k(e) = 0$ for each $k \geq 0$.

Thus $b_m = eb_m$ *and* $a_n x^n b_m x^m = 0$.

Since J_0 is a δ-*ideal* of R, $\delta^k(b_m) \in J_0$ for each $k \geq 0$

and that $b_m \in r_S(J_0[x;\delta])$.

Hence $b_m \in r_R(J_0)$.

Since $f(x)g(x)=0$, we have $a_n b_m = 0$.

First, we will show that $a_i x^i b_j x^j = 0$ for $i = 0, ..., n$ and $j = 0, ..., m$.

Now, let $g(x) = b_0 + ... + b_m x^m \in r_S(J)$ and $f(x) = a_0 + ... + a_n x^n \in J$.

Continuing in this way, we can show that $a_i\ x^i e = 0$ for each $i = 0, ..., n$.

Hence $f(x)e = 0$ and so $eS \subseteq r_S(J)$.

Hence $f(x)e = (a_0 + ... + a_{n-1}x^{n-1})e = ... + a_{n-1}ex^{n-1}$.

Thus $a_{n-1}e \in J_0$, and $a_{n-1}\delta^k(e) = a_{n-1}e\delta^k(e) = 0$ for each $k \geq 0$.

Hence $a_{n-1}x^{n-1}e = 0$.

Let $f(x) = a_0 + ... + a_n x^n \in J$.

Then $a_n \in J_0$

and that $a_n\delta^k(e) = 0$ for each $k \geq 0$.

We claim that $r_S(J) = eS$.

Thus $r_S(J_0[x;\delta]) = eS$. $\quad$ ($\because$ from $eS \subseteq r_S(J_0[x;\delta])$ and $r_S(J_0[x;\delta]) \subseteq eS$)

Since $J_0 e = 0$ and J_0 is δ-*ideal* of R (from J_0 is a δ-*ideal* of R and $r_R(J_0) = eR$)

we have $J_0\delta^k(e) = 0$ for each $k \geq 0$.

Hence $\delta^k(e) = e\,\delta^k(e)$ and $eS \subseteq r_S(J_0[x;\delta])$.

Clearly $r_S(J_0[x;\delta]) \subseteq eS$.

Then J_0 is a δ-*ideal* of R. Since R is δ-Quasi-Baer ring, $r_R(J_0) = eR$ for some

idempotent $e \in R$.

3.3.4 Theorem. Let R be a ring and S = *R[x; δ]*. Then S is $\bar{\delta}$-Quasi-Baer for every extended derivation $\bar{\delta}$ on S of δ.

Proof. Suppose that R is $\bar{\delta}$-Quasi-Baer for every extended derivation $\bar{\delta}$ on S of δ. Let I be any *δ-ideal* of R. Then *I[x; δ]* is $\bar{\delta}$-ideal of S.

Since S is $\bar{\delta}$-Quasi-Baer, $r_S(I[x;\delta]) = e(x)S$ for some idempotent $e(x) \in S$.

Hence $r_S(I[x;\delta]) = e_0 S$ for some idempotent $e_0 \in R$, by Theorem 3.31.

Since $r_R(I) = r_S(I[x;\delta]) \cap R = e_0 R$, R is δ-Quasi-Baer.

3.4 Derivation Polynomials over Quasi-Baer Ring

Three commonly used operations for polynomials are addition " + ", multiplication " . " and composition " ∘ ". Observe that (*R[x]*, +, .) is a ring and (*R[x]*, +, ∘) is a nearring where the composition indicates substitution of *f(x)* into *g(x)*, explicitly $f(x) \circ g(x) = f(g(x))$ for each $f(x), g(x) \in R[x]$.

Let δ is derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \delta a(b), \forall a,b \in R$. Since *R[x; δ]* is an abelian nearring under addition and composition, it is natural to investigate the nearring of derivation polinomials (*R[x; δ]*, +, ∘). We use *R[x; δ]* to denote the nearring of derivation polynomials (*R[x; δ]*, +, ∘) with coefficients from R and $R_0[x;\delta]$ = $\{f \in R[x;\delta] \mid f$ has zero constant term$\}$ the 0-symmetric nearring of *R[x; δ]*.

3.4.1 Example

Let $f(x) = a_0 + a_1 x$

and $g(x) = b_0 + b_1 x + b_2 x^2 \in R[x;\delta]$.

Showing that $f(x) \circ g(x)$!

Answered. Through a simple calculation, we have

$$f(x) \circ g(x) = (f(x))g$$

$$= b_0 + b_1(f(x)) + b_2(f(x))^2$$

$$= (b_0 + b_1a_0 + b_2a_0^2 + b_2a_1\delta(a_0)) + (b_1a_1 + b_2a_0a_1 + b_2a_1a_0 + b_2a_1\delta(a_1))x + b_2a_1^2x^2.$$

3.4.2 Lemma. Let R be a reduced ring and $a, b \in R$. Then we have the following :

1) If $ab = 0$, *then* $a\delta^m(b) = 0 = \delta^m(a)b$ for any positive integers *m*.
2) If $e^2 = e \in R$, *then* $\delta(e) = 0$.

Proof.

(1) It is enough to show that $a\delta(b) = \delta(a)b = 0$.

If *ab = 0*, then $\delta(ab) = \delta(a)b + a\delta(b) = 0$.

Hence $a\delta(a)b + a^2\delta(b) = 0$ ($\because$ right multiplication with *a*, so that *a*.0 = 0)

and that $a\delta(b) = 0$,

since *R* is reduced and *ab = 0*.

(2) If $e^2 = e$, *then* $\delta(e) = \delta(e)e + e\delta(e)$.

Since *R* is reduced, so *e* belong to the center of *R*.

Hence, $2e\delta(e) = e\delta(e)$ and that $e\delta(e) = 0$.

Thus $\delta(e) = 0$.

3.4.3 Lemma. Let δ be a derivation of aring *R* and *R[x; δ]* the nearring of derivation polynomials over *R*. Let *R* be a reduced ring. Then :

1) If $E(x) \in R[x;\ \delta]$ is an idempotent, then $E(x) = e_1 x + e_0$, where e_1 is an idempotent in R with $e_1 e_0 = 0$.

2) $R[x;\ \delta]$ is reduced.

Proof.

(1) Let $E = e_0 + \ldots + e_n x^n$ be an idempotent of $R[x;\ \delta]$.

Since $E(x) \circ E(x) = E(x)$, we have $e_n^{n+1} = 0,\ if\ \ n \geq 2.$

Thus $e_n = 0$, since R is reduced.

Therefore, $E(x) = e_0 + e_1 x$.

Clearly, e_1 is an idempotent of R and $e_1 e_0 = 0$.

(2) Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n \in R[x;\delta]$ such that $f(x) \circ f(x) = 0$.

Then $a_n^{n+1} = 0$. Hence $a_n = 0$, since R is reduced.

We have $a_i = 0$ for each $0 \leq i \leq n.$

Therefore $f(x)=0$ and $R[x;\ \delta]$ is reduced.

3.4.4 Example

Let $R = Z_6\ and\ S = \{2x+2,\ 2x+5\}$. From lemma 3.4.3, all idempotents in $Z_6[x]\ are\ \{0,1,2,3,4,5,x,3x,3x+2,\ 3x+4,4x,4x+3\}$.

Note that $x - c \in r(c)\ and\ x - c \notin r(S)$ for all constant idempotents $c \in Z_6[x]$.

Also, the possible idempotents $E(x) \in Z_6[x]$

such that $r(S) = r(E(x))$ are either $4x\ or\ 4x+3$.

Observe that,

$3x \in r(4x)$ *but* $3x \notin r(S)$, *and also* $3x^3 + 3 \in r(4x+3)$ *but* $3x^3 + 3 \notin r(S)$.

Therefore, there is no idempotent $E(x) \in Z_6[x]$ such that $r(S) = r(E(x))$.

Consequently, $Z_6[x] \notin B_{r2}$.

CHAPTER IV

ENCLOSURE

4.1 Conclusion

According to result of discussion at Chapter III, hence can be taken by conclusion, those are:

1. Let $\delta : R \to R$ is a derivation of R. A ring R is called δ-Quasi-Baer ring if the right annihilator ($r_R(X) = \{c \in R \mid dc = 0, \forall d \in X\}$) of every δ-ideal (an ideal I of R is called *δ-ideal* if $\delta(I) \subseteq I$) of R is generated by an idempotent of R. An idempotent is same with meaning of projection. An idempotent p is called a projection if $p = p^*$ (where * is an involution)
2. Let δ : R → R is derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \delta a(b), \forall a, b \in R$, so that the extension of $\delta : R \to R$ is $\bar{\delta}(ab) = \delta(a)b - \delta a(b), \forall a, b \in R$. For a ring R with a derivation δ, there exists a derivation on $S = R[x;\ \delta]$ which extends δ. Considering an inner derivation $\bar{\delta}$ on S by x defined by $\bar{\delta}(f(x)) = x\,f(x) - f(x)\,x, \forall f(x) \in S$.

   Then,

   $$\bar{\delta}(f(x)) = \delta(a_0) + ... + \delta(a_n)x^n \ \ for\,all\ \ f(x) = a_0 + ... + a_n x^n \in S$$

   $$and\ \bar{\delta}(r) = \delta(r), \forall r \in R$$

which means that $\overline{\delta}$ is an extension of δ. We call such a derivation $\overline{\delta}$ on S an extended derivation of δ. For each $a \in R$ and nonnegative integer n, there exist $t_0, \ldots, t_n \in Z$ *such that* $x^n a = \sum_{i=0}^{n} t_i \delta^{n-i}(a) x^i$.

3. Let $R[x; \delta]$ is the polynomial ring whose elements are the polynomials denote $\sum_{i=0}^{n} r_i x^i \in R, r_i \in R$, where the multiplication operation by $xb = bx + \delta(b), \forall\ b \in R$. Three commonly used operations for polynomials are denoted addition " + ", multiplication " . " and composition " ∘ ". Observe that (*R[x]*, +, .) is a ring and (*R[x]*, +, ∘) is a left nearring where the substitution indicates substitution of *f(x)* into *g(x)*, explicitly *f(x)*∘*g(x)* = *f(g(x)* for each *f(x), g(x)* ∈ *R[x]*.

4.2 Suggestion

At this thesis, the writer only focussed at discussion fundamental of the problem of δ-Quasi-Baer ring. Hence, is suggested at other researcher to perform a research morely about δ-Quasi-Baer ring, with searching natures of others.

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